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ON HIGHER DIMENSIONAL LIGHT BULB THEOREM

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It is well known that any two arcs with the same end points in R^3 or S^3 can be moved isotopically from one to the other keeping the end points fixed during the deformation. This is called Light Bulb Theorem ([8]). In this paper, we prove the Light Bulb Theorem is valid in the higher dimensional case. As an application, we will give a characterization of invertible knots concerning with J. M. Montesinos' question ([4]).

Throughout the paper, we work in the piecewise linear category.

For an oriented manifold X , by $-X$ we mean the manifold with the opposite orientation, and for oriented manifolds X and Y , $X=Y$ means that $X=Y$ as point sets and that the orientations of X and Y coincide. A pair (X, Y) means a pair of manifolds X, Y with $X \supset Y$. In this paper, every submanifold in a manifold is assumed to be locally flat. A ball pair (B^n, B^m) is *unknotted* if it is homeomorphic to $(B^m \times [-1, 1]^{n-m}, B^m \times (0, \dots, 0))$ as a pair. A sphere pair (S^n, S^m) is *unknotted* if S^m bounds an $(m+1)$ -disk in S^n .

A map $F: X \times I \rightarrow Y \times I$ is *level preserving* if, for $t \in I$, $F(X, t) \subset Y \times \{t\}$. For a level preserving map $F: X \times I \rightarrow Y \times I$, we define $F_t: X \rightarrow Y$ by $F_t(x) = F(x, t)$ for $x \in X, t \in I$. An isotopy of X is a level preserving homeomorphism $F: X \times I \rightarrow Y \times I$ such that $F_0 = \text{id}$. By id or 1_X , we mean the identity map of X onto itself. An isotopy F of X is said to *fix* a subset V in X if $F(x, t) = (x, t)$ for $x \in V, t \in I$. A homeomorphism f of X onto itself is *isotopic to the identity* if there exists an isotopy F of X such that $F_1 = f$, and f is *isotopic to the identity keeping V fixed* if F fixed V .

Let $P(n, m)$ and $Q(n, m)$ be the following assertions:

$P(n, m)$: Let (B^n, B^m) be an unknotted ball pair, and f an orientation preserving homeomorphism of B^n onto itself such that $f|_{B^m}$ is the identity map. Then f is isotopic to the identity keeping B^m fixed.

$Q(n, m)$: The same assertion as the above $P(n, m)$ by rewriting an unknotted ball pair (B^n, B^m) as an unknotted sphere pair (S^n, S^m) .

Then we will first prove the following:

THEOREM 1. $P(n, m)$ and $Q(n, m)$ hold for $n > m \geq 0$.

The author should thank to Prof. M. Kato for pointing out that the above theorem is essentially same as Lemma 8 in [7].

For completeness we give a proof of Theorem 1. For this, we give some preliminaries.

Let F be a homeomorphism from $M \times 0 \cup \partial M \times I \cup M \times 1$ onto itself such that $F|_{\partial M \times I}$ is an isotopy of ∂M and that $F|M \times 0$ is the identity map, then we call F a *pre-isotopy* of M .

The following Lemma 1 is trivial by using Alexander's trick, or cone extension:

LEMMA 1. *Let (B^n, B^m) be an unknotted ball pair for $n > m \geq 0$, and F a pre-isotopy of B^n such that $F|_{B^m \times 0}$ and $F|_{B^m \times 1}$ are the identity maps, and that $F|_{\partial B^n \times I}$ fixes ∂B^m . Then F can be extended to an isotopy of B^n which keeps B^m fixed.*

LEMMA 2. *$P(n, 0)$ holds for $n > 0$.*

PROOF. Let f be an orientation preserving homeomorphism of B^n onto itself such that $f(B^0) = B^0$. Then there exists an isotopy F of ∂B^n such that $F_1 = f|_{\partial B^n}$, because any orientation preserving homeomorphism of a sphere onto itself is isotopic to the identity. We can easily extend F to obtain a pre-isotopy, denoted by the same symbol F , of B^n such that $F_1 = f$. By Lemma 1, $P(n, 0)$ is proved.

LEMMA 3. *If $Q(n, m)$ is true, then $P(n+1, m+1)$ holds.*

PROOF. Let (B^{n+1}, B^{m+1}) be an unknotted ball pair, and f an orientation preserving homeomorphism of B^{n+1} onto itself such that $f|_{B^{m+1}}$ is the identity. Then $\partial f = f|_{\partial B^{n+1}}$ is an orientation preserving homeomorphism of S^n onto itself such that $\partial f|_{S^m}$ is the identity, where $S^n = \partial B^{n+1}$ and $S^m = \partial B^{m+1}$. By hypothesis $Q(n, m)$, there exists an isotopy ∂F of S^n such that $\partial F_1 = \partial f$, and ∂F fixes S^m . Then we can construct a pre-isotopy F of B^{n+1} from ∂F such that $F|_{\partial B^{n+1} \times I} = \partial F$, and $F|_{B^{m+1} \times 0}$ and $F|_{B^{m+1} \times 1}$ are the identities. By Lemma 1, $P(n+1, m+1)$ is proved.

LEMMA 4. *If $P(n, m)$ is true, then $Q(n, m)$ holds.*

PROOF. Let (S^n, S^m) be an unknotted sphere pair, and f an orientation preserving homeomorphism such that $f|_{S^m}$ is the identity. Let B_+^m be an m -ball in S^m , $B_-^m = \text{cl}(S^m - B_+^m)$, and B_+^n a relative regular neighborhood of B_+^m mod B_-^m in S^n ([3]). Then $f(B_+^n)$ is a relative regular neighbourhood of B_+^m mod B_-^m in S^n . By the uniqueness of relative regular neighbourhood ([3]), we can find an orientation preserving homeomorphism g of S^n onto itself such that $g(B_+^n) = B_+^n$,

$g|S^m$ is the identity, and f is isotopic to g keeping S^m fixed. Let $g_{\pm} = g|B_{\pm}^n$, where $B_{\pm}^n = \text{cl}(S^n - B_{\mp}^n)$. Then $g_{\pm}$ is an orientation preserving homeomorphism of $B_{\pm}^n$ onto itself such that $g_{\pm}|B_{\pm}^m$ is the identity and $g_{+}|\partial B_{+}^m = g_{-}|\partial B_{-}^m$.

By $P(n, m)$, there exists an isotopy G^{+} of B_{+}^n such that $g_{+} = G_1^{+}$ and G^{+} fixes B_{+}^m . Let G^{-} be a homeomorphism defined by

$$G^{-}(x, t) = \begin{cases} (x, t) & \text{if } x \in B_{-}^n, \quad t = 0, \\ G^{+}(x, t) & \text{if } x \in \partial B_{-}^n, \quad t \in I, \\ (g_{-}(x), t) & \text{if } x \in B_{-}^n, \quad t = 1. \end{cases}$$

Then G^{-} is an pre-isotopy of B_{-}^n , hence by Lemma 1, G^{-} can be extended to an isotopy, denoted by the same symbol G^{-} , of B_{-}^n keeping B_{-}^m fixed. By G^{+} and G^{-} , g is isotopic to the identity keeping S^m fixed. Thus f is isotopic to the identity keeping S^m fixed. This completes the proof of Lemma 4.

PROOF OF THEOREM 1. Theorem 1 is trivially followed from Lemmas 2, 3 and 4.

The following is directly implied by Theorem 3 in [2]:

PROPOSITION 1. *Let M be a locally flat submanifold in the interior of a manifold Q , and F an isotopy of M . Then there exists an isotopy H of Q such that the following is commutative:*

$$\begin{array}{ccc} Q \times I & \xrightarrow{H} & Q \times I \\ \uparrow i \times 1_I & & \uparrow i \times 1_I \\ M \times I & \xrightarrow{F} & M \times I \end{array}$$

where i is the inclusion map of M into Q .

LEMMA 5. *Let B_0^m and B_1^m be m -disks in S^n for $n > m \geq 0$ such that $\partial B_0^m = \partial B_1^m$. Then there exists an orientation preserving homeomorphism f of S^n onto itself such that $f(B_0^m) = B_1^m$ and $f|\partial B_0^m = 1_{\partial B_0^m}$.*

PROOF. By choosing suitable orientations of B_0^m and B_1^m , we can get an orientation preserving homeomorphism g of S_n onto itself such that $g|B_0^m: B_0^m \rightarrow B_1^m$ and $g|\partial B_1^m: \partial B_1^m \rightarrow \partial B_1^m$ are orientation preserving homeomorphisms. Then there exists an isotopy G of ∂B_1^m such that $G_1 = (g|\partial B_1^m)^{-1}$. By the parallel argument to Lemma 1, i.e. by cone extension, we can get an isotopy F of B_1^m such that $F_1|\partial B_1^m = G_1$. By Proposition 1, there exists an isotopy H of S^n such that the following commutes:

$$\begin{array}{ccc}
S^n \times I & \xrightarrow{H} & S^n \times I \\
\uparrow i \times 1_I & & \uparrow i \times 1_I \\
B_1^m \times I & \xrightarrow{F} & B_1^m \times I
\end{array}$$

where i is an inclusion map of B_1^m into S^n . Let $f = H_1 \circ g$, then we can easily check that $f(B_0^m) = B_0^m$ and $f|_{\partial B_1^m} = 1_{\partial B_1^m}$. This completes the proof of Lemma 5.

Combining Theorem 1 and Lemma 5, we have the following Higher Dimensional Light Bulb Theorem:

THEOREM 2. *Let B_0^m and B_1^m be locally flat m -disks in S^n for $n > m \geq 0$ such that $\partial B_0^m = \partial B_1^m$. Then there exists an isotopy of S^n which moves B_0^m onto B_1^m keeping ∂B_0^m fixed.*

Without the assumption of locally flatness, W. B. R. Lickorish proved Theorem 2 in the case of $n - m \geq 3$ ([5]).

In the rest of the paper, every manifold is assumed to be oriented. A knot K^n is said to be *invertible* if there exists an orientation preserving homeomorphism f of S^{n+2} onto itself such that $f(K^n) = -K^n$. For an unknotted sphere pair (S^n, S^{n-1}) , f is a *reflection of S^n with respect to S^{n-1}* if f is an orientation reversing homeomorphism of S^n onto itself such that $f|_{S^{n-1}}$ is the identity map of S^{n-1} onto itself.

In [6], J. M. Montesinos asked wheather each invertible knot is strongly invertible or not [4]. Concerning this question, counter examples are given by W. Whitten ([9]).

The following Theorem 3 shows that each invertible knot has a "weakened" strong invertibility.

THEOREM 3. *Let K^n be an invertible knot in S^{n+2} . Then there exists an isotopy F of S^{n+2} such that*

- (1) $\text{Fix}(F) \cap K^n$ is an unknotted $(n-1)$ -sphere in K^n , where $\text{Fix}(F) = \{x \in S^{n+2} | F_t(x) = x \text{ for } t \in I\}$,
- (2) $F_1|_{K^n}$ is a reflection of K^n with respect to $\text{Fix}(F) \cap K^n$.

To prove Theorem 3, we will give some lemmas.

LEMMA 6. *Let S^m be in S^n , and S_0^k and S_1^k be unknotted oriented k -spheres in S^m ($n > m > k \geq 0$). Then there exists an isotopy F of S^n such that $F_t(S^m) = S^m$ for $t \in I$, and $F_1(S_0^k) = S_1^k$.*

PROOF. By [1], there exists an isotopy of S^m which sends S_0^k to S_1^k . From this fact and Proposition 1, the required isotopy is easily obtained.

By the similar way, the following is easily proved, so we omit the proof.

LEMMA 7. Let S^n , S^m and S^k be spheres with $S^k \subset S^m \subset S^n$, and f be an orientation preserving homeomorphism of S^k onto itself. Then there exists an isotopy F of S^n such that $F_t(S^m) = S^m$ for $t \in I$, and $F_1 \circ f$ is the identity map of S^k onto itself.

PROOF OF THEOREM 3. Let B_0 be an n -disk in K^n , and $B_1 = -\text{cl}(K^n - B_0)$, hence $K^n = B_0 \cup (-B_1)$. By the invertibility of K^n , there exists an orientation preserving homeomorphism f of S^{n+2} onto itself with $f(K^n) = -K^n$. By Lemma 6, there exists an orientation preserving homeomorphism g of S^{n+2} such that $g(K^n) = K^n$ and $g(f(\partial B_0)) = \partial B_0$. By considering orientations, $g \circ f(B_0) = B_1$ and $g \circ f(B_1) = B_0$. Thus $(g \circ f)|_{\partial B_0}$ is an orientation preserving homeomorphism of ∂B_0 onto itself. Then, by Lemma 7, there exists an orientation preserving homeomorphism h of S^{n+2} onto itself such that $h(K^n) = K^n$ and $(h \circ g \circ f)|_{\partial B_0}$ is the identity. Let $\phi = h \circ g \circ f$, then ϕ is an orientation preserving homeomorphism of S^{n+2} onto itself such that $\phi(K^n) = -K^n$ and $\phi|_{\partial B_0}$ is the identity, hence $\phi|_{K^n}$ is a reflection of K^n with respect to ∂B_0 . By Theorem 1, we can obtain the required isotopy.

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