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(Citation)
Kobe Journal of Mathematics,3(2):209-212

(Issue Date)
1987-02

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0034646

(URL)
https://hdl.handle.net/20.500.14094/E0034646

NAKANISHI, Y.
KOBE J. MATH.,
3 (1986), 209-212

# A REMARK ON CRITICAL POINTS OF LINK COBORDISM

By Yasutaka NAKANISHI

*Dedicated to the memory of Professor Takehiko Miyata*

(Received May 1, 1986)

We consider a compact oriented surface $F$ smoothly and properly embedded in $R^3 \times I$ which is the direct product space of the Euclidean 3-space $R^3$ and the unit interval $I=[0, 1]$. Note that $F$ is ambient isotopic to a surface which has only *elementary critical points* (see, [4], I). So, we can assume that $F$ has only elementary critical points, and let $c(F)$ be the number of them.

In this note we will estimate $c(F)$ by numerical invariants of links $(L_i, R^3)=(F \cap R^3 \times \{i\}, R^3 \times \{i\})$ $(i=0, 1)$. Let $\mu(L)$ be the number of components of a link $L$. We can easily see the following fact:

$$c(F) \geqq |\mu(L_0)-\mu(L_1)|, \quad \text{and}$$
$$c(F) \equiv \mu(L_0) - \mu(L_1) \pmod 2,$$

which are rough but best possible.

For an oriented link $(L, R^3)$, let $X(L)=R^3-L$, and take the unique infinite cyclic covering $p\colon \tilde{X}(L) \rightarrow X(L)$ associated with the epimorphism $\gamma\colon \pi_1(X(L)) \rightarrow \langle t \rangle$ sending each meridian of $L$ to $t$, where $\langle t \rangle$ is the infinite cyclic group generated by a letter $t$. The 1-dimensional integral homology group $H_1(\tilde{X}(L); \boldsymbol{Z})$ forms a finitely generated $\boldsymbol{Z}\langle t \rangle$-module, and we let $m(L)$ be the minimum size of such matrices [2]. By convention, $m(L)=0$ iff $H_1(\tilde{X}(L); \boldsymbol{Z})$ is trivial.
For an oriented link $(L, R^3)$, we can define invariants of a link: the signature $\sigma(L)$ and the nullity $n(L)$ in the sense of Murasugi [5]. The following Theorem is an answer to a question of Professor A. Kawauchi at a KOOK seminar.

THEOREM. *For the above $F$, $L_0$, and $L_1$, we have*

$$c(F) \geqq |m(L_0)-m(L_1)|, \quad \textit{and}$$
$$c(F) \geqq |\sigma(L_0)-\sigma(L_1)| + |n(L_0)-n(L_1)|.$$

PROOF. If $F$ has minimal points and/or maximal points, then we can

* Supported by the Japan Society for the Promotion of Science.

construct a new surface $F'$ in $R^3 \times [0, 1]$, which has $c(F)$ saddle points, no minimal points, and no maximal points, by piping as shown in Fig. 1 (cf. [4], II).

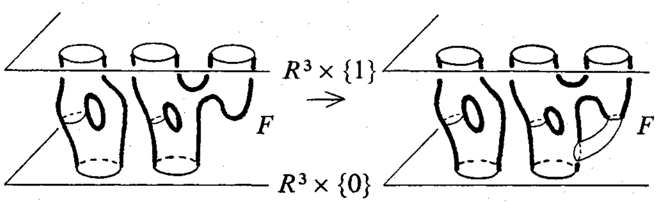


Fig. 1

Furthermore, we can assume that the levels of such saddle points are mutually distinct i.e. for the integer $k=c(F)$, we can choose numbers $t(i)$'s $(i=0, 1, 2,\dots, k)$ with $0=t(0)<t(1)<t(2)<\cdots<t(k)=1$ such that $F \cap R^3 \times [t(i), t(i+1)]$ has only one saddle point, and $F$ has no saddle point at $R^3 \times \{t(i)\}$ $(i=0, 1, 2,\dots, k)$.

We denote the link $(F \cap R^3 \times \{t(i)\}, R^3 \times \{t(i)\})$ by $(L(i), R^3)$. From the following inequalities:

$$\begin{aligned}
|m(L_0)-m(L_1)| &\leqq |m(L(0))-m(L(1))| + |m(L(1))-m(L(2))| \\
&\quad +\cdots+ |m(L(k-1))-m(L(k))|, \\
|\sigma(L_0)-\sigma(L_1)| &\leqq |\sigma(L(0))-\sigma(L(1))| + |\sigma(L(1))-\sigma(L(2))| \\
&\quad +\cdots+ |\sigma(L(k-1))-\sigma(L(k))|, \quad \text{and} \\
|n(L_0)-n(L_1)| &\leqq |n(L(0))-n(L(1))| + |n(L(1))-n(L(2))| \\
&\quad +\cdots+ |n(L(k-1))-n(L(k))|,
\end{aligned}$$

it is sufficient to show Theorem for the case that $F$ has only one saddle point.

From now, we assume that $F$ has only one saddle point, no minimal point, and no maximal point; in other words, $L_1$ is obtained from $L_0$ by only one fusion (and conversely, $L_0$ is obtained from $L_1$ by only one fission). Let $\beta_0$ be a band which realizes the above fusion on $L_0$ and $S_0$ a connected Seifert surface of $L_0$.

By Yanagawa's method in [8], we can construct a connected Seifert surface $S_1$ of $L_1$ from $S_0$ and $\beta_0$ by piping along $\beta_0$ to kill singularities of $S_0 \cap \beta_0$. Now, we choose simple loops $\{a_i\}$ which represent a basis of $H_1(S_1-\beta_0; \boldsymbol{Z})$ as a free abelian group, and we further choose a simple loop $b_0$ rounding once along $\beta_0$ such that $\{a_i\} \cup b_0$ represent a basis of $H_1(S_1; \boldsymbol{Z})$ as a free abelian group.
We obtain $V_1=\left|\begin{array}{c:c} lk(a_i, a_j^+) & lk(a_i, b_0^+) \\ \hdashline lk(b_0, a_j^+) & lk(b_0, b_0^+) \end{array}\right|$ for a Seifert matrix of $L_1$. Therefore, the 1-dimensional integral homology group $H_1(\tilde{X}(L_1); \boldsymbol{Z})$ has a square presentation matrix $V-t\cdot V'$ as a $\boldsymbol{Z}\langle t\rangle$-module, where $V'$ denotes the transposed matrix of $V$, which we rewright out

$$\left| \begin{array}{c:c} lk(a_i, a_j^\dagger) - t \cdot lk(a_j, a_i^\dagger) & lk(a_i, b_0^\dagger) - t \cdot lk(b_0, a_i^\dagger) \\ \hdashline lk(b_0, a_j^\dagger) - t \cdot lk(a_j, b_0^\dagger) & lk(b_0, b_0^\dagger) - t \cdot lk(b_0, b_0^\dagger) \end{array} \right| .$$

Notice that $(S_1 - \beta_0) \cup (S_1 \cap \beta_0)$ is a connected Seifert surface of $L_0$. Therefore, the sub-matrix $|lk(a_i, a_j^\dagger)|$ is a Seifert matrix of $L_0$, and the sub-matrix $|lk(a_i, a_j^\dagger) - t \cdot lk(a_j, a_i^\dagger)|$ is a square presentation matrix of $H_1(\tilde{X}(L_0); \boldsymbol{Z})$ as a $\boldsymbol{Z}\langle t \rangle$-module. By the definition, this sub-matrix can be reduced to a square matrix of size $m(L_0)$ by a finite sequence of fundamental transformations of presentation matrices of $\boldsymbol{Z}\langle t \rangle$-modules. Those transformations can be applied to $V - t \cdot V'$ regardless of coefficients of the last row and the last column. Therefore, $V - t \cdot V'$ can be reduced to a square matrix of size $m(L_0) + 1$.

Hence, we have $m(L_1) \leqq m(L_0) + 1$.

Conversely, from a connected Seifert surfaces $S_1^*$ of $L_1$ and a band $\beta_1$ which realizes the fission on $L_1$, we can construct a connected Seifert surfaces $S_0^*$ of $L_0$. By parallel argument to the above, we have $m(L_0) \leqq m(L_1) + 1$.

Hence, we have $|m(L_0) - m(L_1)| \leqq 1$, and the proof of the first part is complete.

Concerning the second part, Murasugi have already shown that $|\sigma(L_0) - \sigma(L_1)| + |n(L_0) - n(L_1)| = 1$ if $L_1$ is obtained by only one fusion from $L_0$ in [5, Lemma 7.1].

Hence, the proof is complete.

COROLLARY. *Let $F$ be a compact oriented surface which is smoothly and properly embedded in $R^3 \times [0, \infty)$, and which has only elementary critical points. For $(L, R^3) = (F \cap R^3 \times \{0\}, R^3 \times \{0\})$, we have*

$$c(F) \geqq m(L) + 1, \quad \text{and}$$

$$c(F) \geqq |\sigma(L)| + |n(L) - 1| + 1.$$

PROOF. From the compactness of $F$, $F$ can be assumed to be in $R^3 \times [0, s)$ for a sufficiently large number $s$. Then we can construct a new surface $F'$ from $F$ by piping from one maximal point of $F$ to a trivial knot $O$ in $R^3 \times \{s\}$ as shwon in Fig. 2.

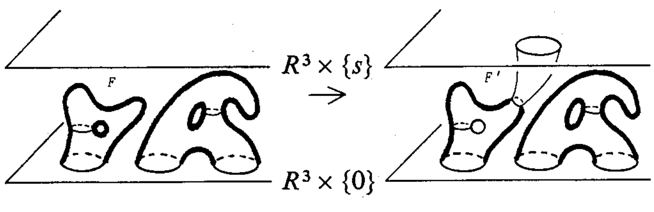


Fig. 2

By Theorem, we have $c(F')\geqq|m(L)-m(O)|$, and $c(F')\geqq|\sigma(L)-\sigma(O)|+|n(L)-n(O)|$. From $m(O)=\sigma(O)=0$, and $n(O)=1$, we have the required inequality by $c(F)-1=c(F')$. Hence, the proof is complete.

REMARK. The first estimate of Corollary: $c(F)\geqq m(L)+1$, was shown for the case $L$ is a knot in [6], [7], and [1].

After my work, Kawauchi [2] gave a sharp estimate: $c(F)-2c_0(F)\geqq m(L)+1$, by delicate algebraic argument on modules, where $c_0(F)$ is the number of the minimal points of $F$.

Concerning the second estimate, Murasugi [5] took an interest in an estimate of genus $h^*(F)$ of a surface $F$, and he gave $|\sigma(L)|\leqq 2h^*(F)+\mu(L)-1$.

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