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ENUMERATION AND REPRESENTATION OF NONISOMORPHIC WEIGHTED GRAPHS

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1. Introduction

Let $X=\{1, 2,..., k\}$, where k is an integer with $k\geq 2$, and let $X^{(2)}$ be the set of all 2-subsets of X. An element $\{p, q\}$ of $X^{(2)}$ is simply written by pq. Suppose that a number t is assigned to an element pq in $X^{(2)}$. Then the number t is called the weight of pq. For a positive integer l, consider a weighted graph with the point-set X in which each $pq\in X^{(2)}$ has an integral weight ω_{pq} satisfying $0\leq\omega_{pq}\leq l-1$. Clearly, $\omega_{pq}=\omega_{qp}$, since a graph treated here is an undirected graph. A weighted graph is a multigraph if each weight is regarded as the number of multiple lines. In particular, $\omega_{pq}=0$ means that the points p and q are not adjacent. When $l=2$, a weighted graph is equivalent to an ordinary undirected graph (simple graph). Two weighted graphs are said to be isomorphic if there is a one-to-one correspondence from the point-set of one graph onto that of the other graph which preserves weights assigned to the pairs of points. For each pair (l, k), we want to determine the number of nonisomorphic weighted graphs and obtain their representatives. The enumeration of simple graphs has been established and the representatives have been obtained for some k (e.g., see Appendix I of Harary [1] and Appendix I of Harary and Palmer [2]).

Let $\boldsymbol{\omega}$ be a $\binom{k}{2}$-rowed vector whose i-th element is given by ω_{pq}, where p and q $(1\leq p<q\leq k)$ are obtained by $i=(p-1)(2k-p)/2+(q-p)$. Further, let $\Omega=\{\boldsymbol{\omega}=(\omega_{12}, \omega_{13},..., \omega_{k-1,k})|\omega_{pq}=0, 1,..., l-1;\ 1\leq p<q\leq k\}$. It is easily seen that there is a one-to-one correspondence between the $\boldsymbol{\omega}$'s and the weighted graphs. Therefore the total number of weighted graphs is the same as the cardinality of Ω (i.e., $|\Omega|=l^{\binom{k}{2}}$). Let S_k be a symmetric group of degree k on the X and let $S_k^{(2)}$ be the permutation group induced by S_k which acts on $X^{(2)}$. Each permutation σ in S_k induces a permutation σ' in $S_k^{(2)}$ such that for every element pq in $X^{(2)}$,

$$\sigma'\{p, q\} = \{\sigma p, \sigma q\}.$$

For $\sigma\in S_k$ and $\boldsymbol{\omega}=(\omega_{12}, \omega_{13},..., \omega_{k-1,k})\in\Omega$, define $\boldsymbol{\omega}^{\sigma}=(\omega_{\sigma'(12)}, \omega_{\sigma'(13)},...,$

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$\omega_{\sigma'(k-1,k)})$, by which σ acts on Ω since $\boldsymbol{\omega}^\sigma \in \Omega$. We can then introduce an equivalence relation as follows. Two elements $\boldsymbol{\omega}$ and $\tilde{\boldsymbol{\omega}}$ in Ω are equivalent, $\boldsymbol{\omega} \sim \tilde{\boldsymbol{\omega}}$ (mod S_k), if there exists a permutation $\sigma \in S_k$ such that $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}^\sigma$ holds. This means that the set Ω is partitioned by equivalence classes, i.e.,

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_R, \tag{1.1}$$

where $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2 \in \Omega_i$ if and only if $\boldsymbol{\omega}_1 \sim \boldsymbol{\omega}_2$ (mod S_k), and $\Omega_i \cap \Omega_{i'} = \phi$ $(i \neq i')$. Here Ω_i's are said to be orbits of S_k. Clearly, the number of nonisomorphic weighted graphs is equal to R, that of orbits of S_k. In Section 2, we give the enumeration of weighted graphs, that is, the number R of orbits of S_k. Essentially, for nonnegative integers $\lambda_0, \lambda_1, \ldots, \lambda_{l-1}$ satisfying $\sum_{t=0}^{l-1} \lambda_t = \binom{k}{2}$, we give the enumeration of weighted graphs in which the number of elements of $X^{(2)}$ having weight t is λ_t for $t = 0, 1, \ldots, l-1$. This gives the number of orbits in a subset $\Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$ of Ω such that exactly λ_t components of $\boldsymbol{\omega} \in \Omega$ have weight t for $t = 0, 1, \ldots, l-1$. These enumerations may be achieved by using Pólya enumeration theorem (see, e.g., [2] or Redfield [3]). However, the methods include complicated and enormous calculations. The method given in this paper is efficient in a sense that calculations are considerably simplified.

When k is even, the enumeration and representation of a weighted graph play an important role in the theory of a statistical search design introduced by Srivastava [4]. In particular, those for $l = 2, 3$ and for k not so large are more important from the point of view of practical usages. The details will be seen in a suceeding paper. The representatives of weighted graphs for $l = 3$ and $k = 4$ are given in Section 3.

2. Enumeration of orbits in Ω

We first state the well-known Burnside's Lemma, which is an important tool in calculating the number of orbits of a group A as it acts on a set Θ.

THEOREM 2.1 (Burnside's Lemma). *The number $N(A)$ of the orbits is given by*

$$N(A) = \frac{1}{|A|} \sum_{\sigma \in A} |\{x \in \Theta;\ x^\sigma = x\}|.$$

Suppose $A = S_k$ and $\Theta = \Omega$. Then $|S_k| = k!$. We here calculate $|\{\boldsymbol{\omega} \in \Omega;\ \boldsymbol{\omega}^\sigma = \boldsymbol{\omega}\}|$ for each $\sigma \in S_k$. Let $j_r(\sigma)$, $r = 1, 2, \ldots, k$, be the number of cycles of length r in the disjoint cycle decomposition of σ. Then $(j_1(\sigma), j_2(\sigma), \ldots, j_k(\sigma))$ is called the type of σ. Of course, $\sum_{r=1}^{k} r j_r(\sigma) = k$. Without confusion on σ, we simply write $j_r = j_r(\sigma)$ for $r = 1, \ldots, k$. The following lemma can easily be shown by [2, p. 84]:

LEMMA 2.2. *Let $\sigma \in S_k$ be of type $(j_1, j_2,\dots, j_k)$. In the disjoint cycle decomposition of $\sigma' \in S_k^{(2)}$ induced by σ, the number of cycles of σ', $\mu(j_1, j_2,\dots, j_k)$, is then given by*

$$\mu(j_1, j_2,\dots, j_k) = \sum_{r=1}^{k} \left\{\left[\frac{r}{2}\right]j_r + r\binom{j_r}{2}\right\} + \sum_{1\le r<r'\le k}\sum (r, r')j_r j_{r'},$$

where (r, r') denotes the greatest common divisor of r and r', and $[\cdot]$ denotes the Gauss notation. Furthermore, among those cycles, $\left\{r\binom{j_r}{2}+\left[\frac{r-1}{2}\right]j_r+\delta_r j_{2r}\right\}$ cycles have length r, $(1\le r\le k)$, and $(r, r')j_r j_{r'}$ cycles have length $[r, r']$, $(1\le r<r'\le k)$, where $\delta_r=1$ for $1\le r\le[k/2]$ and $\delta_r=0$ for $[k/2]<r\le k$, and $[r, r']$ denotes the least common multiple.

LEMMA 2.3. *Let $\sigma \in S_k$ be of type $(j_1, j_2,\dots, j_k)$. Then*

$$|\{\boldsymbol{\omega}\in\Omega;\ \boldsymbol{\omega}^\sigma=\boldsymbol{\omega}\}| = l^{\mu(j_1,j_2,\dots,j_k)}. \tag{2.1}$$

PROOF. Suppose $\boldsymbol{\omega}$ is fixed by σ, i.e., $\boldsymbol{\omega}^\sigma=\boldsymbol{\omega}$. Then $\omega_{\sigma'(pq)}=\omega_{pq}$ for all $pq\in X^{(2)}$. Thus the elements of $\boldsymbol{\omega}$ must be constant on each cycle of σ'. Conversely, every vector whose elements are constant on each cycle of σ' is fixed by σ. Since the number of cycles of σ' in the disjoint cycle decomposition is $\mu(j_1, j_2,\dots, j_k)$ of Lemma 2.2, the number of vectors of Ω which are fixed by σ is $l^{\mu(j_1,j_2,\dots,j_k)}$. Hence we have (2.1).

It is well-known that the number of permutations of S_k of type $(j_1, j_2,\dots, j_k)$, $h(j_1, j_2,\dots, j_k)$, is given by

$$h(j_1, j_2,\dots, j_k) = \frac{k!}{(j_1!)(2^{j_2}j_2!)\cdots(k^{j_k}j_k!)}. \tag{2.2}$$

From Theorem 2.1 and Lemma 2.3, we have

THEOREM 2.4. *The number of the orbits in Ω is given by*

$$R = \frac{1}{k!}\sum_{(j_1,\dots,j_k)} h(j_1, j_2,\dots, j_k)l^{\mu(j_1,j_2,\dots,j_k)}, \tag{2.3}$$

where the summation runs over all possible k-plets of nonnegative integers, $(j_1, j_2,\dots, j_k)$, satisfying $\sum_{r=1}^{k} rj_r=k$.

In Table 1, the values of R are given for $l=2$, 3, 4 and for values of k $(3\le k\le 6)$. As mentioned earlier, for $l=2$, the number of R is the same as the number of nonisomorphic simple graphs with k points, which has been given for $1\le k\le 9$ in Table A1 of [2].

However, it is, in general, difficult to find a representative of each orbit of S_k. We need to consider a range as narrow as possible in which we can find a representative. Suppose $\lambda_t(\boldsymbol{\omega})$ is the number of elements ω_{pq} in $\boldsymbol{\omega}\in\Omega$ such that $\omega_{pq}=t$ for $t=0, 1,\dots, l-1$. Denote

Table 1

l \ k	3	4	5	6
2	4	11	34	156
3	10	66	792	25506
4	20	276	10688	1601952

$$\Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}} = \{\boldsymbol{\omega} \in \Omega;\ \lambda_t(\boldsymbol{\omega})=\lambda_t,\ t=0, 1,\dots, l-1\},$$

where λ_t's are nonnegative integers satisfying $\sum_{t=0}^{l-1}\lambda_t=\binom{k}{2}$. If $(\lambda_0, \lambda_1,\dots, \lambda_{l-1}) \neq (\lambda'_0, \lambda'_1,\dots, \lambda'_{l-1})$, then for any $\boldsymbol{\omega} \in \Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$ and $\tilde{\boldsymbol{\omega}} \in \Omega_{\lambda'_0\lambda'_1\cdots\lambda'_{l-1}}$, $\tilde{\boldsymbol{\omega}}$ is not equivalent to $\boldsymbol{\omega}$ under S_k. This means that $\Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$ is given as the union of some orbits in (1.1). For any $\sigma \in S_k$, if $\boldsymbol{\omega} \in \Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$, then $\boldsymbol{\omega}^\sigma \in \Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$, by which σ is also acting on $\Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$.

Consider $\sigma \in S_k$ of type $(j_1, j_2,\dots, j_k)$ and let $z_1, z_2,\dots, z_\mu$ be the list of cycles in the disjoint cycle decomposition of $\sigma' \in S_k^{(2)}$, which is induced by σ, where $\mu=\mu(j_1, j_2,\dots, j_k)$. Further suppose $p(j_1,\dots, j_k;\ \lambda_0,\dots, \lambda_{l-1})$ is the number of maps ψ from $\{z_1, z_2,\dots, z_\mu\}$ to $\{0, 1,\dots, l-1\}$ satisfying

$$\lambda_t = \sum_{z\in\psi^{-1}(t)} L(z), \quad t = 0, 1,\dots, l-1, \tag{2.4}$$

where $L(z)$ is the length of cycle z and $\psi^{-1}(t)=\{z|\psi(z)=t\}$. The following theorem is established:

THEOREM 2.5. *The number of orbits in $\Omega_{\lambda_0\cdots\lambda_{l-1}}$ is given by*

$$R_{\lambda_0,\dots,\lambda_{l-1}} = \frac{1}{k!}\sum_{(j_1,\dots,j_k)} h(j_1,\dots, j_k)p(j_1,\dots, j_k;\ \lambda_0,\dots, \lambda_{l-1}). \tag{2.5}$$

PROOF. Let $\sigma \in S_k$ be a permutation of type $(j_1, j_2,\dots, j_k)$ and consider a vector $\boldsymbol{\omega}$ of $\Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}}$. As seen in the proof of Lemma 2.3, $\boldsymbol{\omega}$ is fixed by σ if and only if the elements of $\boldsymbol{\omega}$ are constant on each cycle in the disjoint cycle decomposition of σ'. Thus it follows that $\boldsymbol{\omega}$ is fixed by σ if and only if $\lambda_t=\lambda_t(\boldsymbol{\omega})$, $t=0, 1,\dots, l-1$, satisfies (2.4). Therefore

$$|\{\boldsymbol{\omega} \in \Omega_{\lambda_0\lambda_1\cdots\lambda_{l-1}};\ \boldsymbol{\omega}^\sigma=\boldsymbol{\omega}\}| = p(j_1,\dots, j_k;\ \lambda_0,\dots, \lambda_{l-1})$$

holds. Hence from Theorem 2.1 we have (2.5).

Clearly, if $\{\lambda_0,\dots, \lambda_{l-1}\}=\{\lambda'_0,\dots, \lambda'_{l-1}\}$ as sets of l elements, then

$$p(j_1,\dots, j_k;\ \lambda_0,\dots, \lambda_{l-1}) = p(j_1,\dots, j_k;\ \lambda'_0,\dots, \lambda'_{l-1}).$$

Therefore we have

COROLLARY 2.6. *Let $\lambda_0,\dots, \lambda_{l-1}$ and $\lambda'_0,\dots, \lambda'_{l-1}$ be nonnegative integers* *satisfying* $\sum_{t=0}^{l-1}\lambda_t=\sum_{t=0}^{l-1}\lambda'_t=\binom{k}{2}$. *If* $\{\lambda_0,\dots,\lambda_{l-1}\}=\{\lambda'_0,\dots,\lambda'_{l-1}\}$, *then*

$$R_{\lambda_0,\dots,\lambda_{l-1}} = R_{\lambda'_0,\dots,\lambda'_{l-1}}.$$

Denote the number of distinct $L(z_q)$'s by d. Let $a_1, a_2,\dots, a_d$ be the list of distinct values of $L(z_1), L(z_2),\dots, L(z_\mu)$, and let f_i be the number of z_q's satisfying $L(z_q)=a_i$, $i=1, 2,\dots, d$. Then clearly, $\sum_{i=1}^{d} f_i=\mu(j_1,\dots,j_k)$ and $\sum_{i=1}^{d} f_i a_i=\sum_{q=1}^{\mu} L(z_q)=\binom{k}{2}$. We denote by Γ the set of all possible d-plets $(\boldsymbol{r}_1, \boldsymbol{r}_2,\dots, \boldsymbol{r}_d)$ of vectors $\boldsymbol{r}_i=(r_i^0, r_i^1,\dots, r_i^{l-1})$, where r_i^t's are nonnegative integers satisfying the following equations:

$$\sum_{t=0}^{l-1} r_i^t = f_i, \quad i = 1, 2,\dots, d, \tag{2.6}$$

$$\sum_{i=1}^{d} a_i r_i^t = \lambda_t, \quad t = 0, 1,\dots, l-2.$$

The following theorem can easily be shown:

THEOREM 2.7. *The number* $p(j_1,\dots,j_k;\ \lambda_0,\dots,\lambda_{l-1})$ *is explicitly written as*

$$p(j_1,\dots,j_k;\ \lambda_0,\dots,\lambda_{l-1}) = \begin{cases} \sum_{\Gamma} \prod_{i=1}^{d} \dfrac{f_i!}{r_i^0!\, r_i^1! \cdots r_i^{l-1}!} & (\text{if } \Gamma\neq\phi) \\ 0 & (\text{if } \Gamma=\phi), \end{cases} \tag{2.7}$$

where $\sum_{\Gamma}$ *denotes the summation over all* $(\boldsymbol{r}_1, \boldsymbol{r}_2,\dots, \boldsymbol{r}_d)$ *belonging to* Γ.

3. Weighted graphs for $l=3$ and $k=4$

We consider the case of $k=4$. By using Lemma 2.2 and (2.2) we obtain the following table:

j_1	j_2	j_3	j_4	$h(j_1,\dots,j_4)$	$\mu=\mu(j_1,\dots,j_4)$	$(L(z_1),\dots, L(z_\mu))$
4	0	0	0	1	6	(1, 1, 1, 1, 1, 1)
2	1	0	0	6	4	(1, 1, 2, 2)
1	0	1	0	8	2	(3, 3)
0	0	0	1	6	2	(2, 4)
0	2	0	0	3	4	(1, 1, 2, 2)

By (2.3), R is 11 and 66 in the respective cases of $l=2$ and 3. These results are seen in Table 1. Consider, for example, $(j_1, j_2, j_3, j_4)=(2, 1, 0, 0)$ or $(0, 2, 0, 0)$. Then $(L(z_1), L(z_2), L(z_3), L(z_4))=(1, 1, 2, 2)$. This gives $d=2$, $a_1=1$, $a_2=2$ and $f_1=f_2=2$.

Suppose $l=3$. From (2.7) we have $p(2, 1, 0, 0;\ 2, 2, 2)=p(0, 2, 0, 0;\ 2, 2, 2)$

$=6$ for $\lambda_0=\lambda_1=\lambda_2=2$, since $\Gamma=\{((2, 0, 0), (0, 1, 1)), ((0, 2, 0), (1, 0, 1)), ((0, 0, 2), (1, 1, 0))\}$. Certainly, there are six ψ maps given as follows:

ψ_1	ψ_2	ψ_3	ψ_4	ψ_5	ψ_6
$z_1 \to 0$	$z_1 \to 0$	$z_1 \to 1$	$z_1 \to 1$	$z_1 \to 2$	$z_1 \to 2$
$z_2 \to 0$	$z_2 \to 0$	$z_2 \to 1$	$z_2 \to 1$	$z_2 \to 2$	$z_2 \to 2$
$z_3 \to 1$	$z_3 \to 2$	$z_3 \to 0$	$z_3 \to 2$	$z_3 \to 0$	$z_3 \to 1$
$z_4 \to 2$	$z_4 \to 1$	$z_4 \to 2$	$z_4 \to 0$	$z_4 \to 1$	$z_4 \to 0$

Obviously, $p(4, 0, 0, 0; 2, 2, 2)=90$. For the remaining cases, $p(j_1,\dots, j_4; 2, 2, 2)=0$, since $L(z_2)>2$ and $\Gamma=\phi$. Therefore, from Theorem 2.5 we have $R_{2,2,2}=(1\cdot 90+6\cdot 6+3\cdot 6)/4!=6$. Similarly $R_{6,0,0}=R_{5,1,0}=1$, $R_{4,2,0}=R_{4,1,1}=2$, $R_{3,3,0}=3$ and $R_{3,2,1}=4$. Note that $R_{\lambda_0,\lambda_1,\lambda_2}=R_{\lambda'_0,\lambda'_1,\lambda'_2}$ if $\{\lambda_0, \lambda_1, \lambda_2\}=\{\lambda'_0, \lambda'_1, \lambda'_2\}$, as stated in Corollary 2.6.

Table 2

ω						ω						ω					
12	13	14	23	24	34	12	13	14	23	24	34	12	13	14	23	24	34
0	0	0	0	0	0	0	0	2	2	0	2	0	2	1	2	1	1
0	0	0	0	0	1	0	0	1	1	1	1	0	2	2	1	1	1
0	0	0	0	0	2	0	1	1	1	1	0	0	0	2	1	1	2
0	0	0	0	1	1	0	0	1	2	1	1	0	1	2	2	1	1
0	0	1	1	0	0	0	0	2	1	1	1	0	1	2	1	2	2
0	0	0	0	1	2	0	0	1	1	2	1	0	1	1	2	2	2
0	0	1	2	0	0	0	1	1	2	1	0	0	2	1	2	2	1
0	0	0	0	2	2	0	0	2	2	1	1	0	2	1	1	2	2
0	0	2	2	0	0	0	0	1	1	2	2	0	1	2	2	2	2
0	0	0	1	1	1	0	1	1	2	2	0	0	2	2	2	2	1
0	0	1	0	1	1	0	0	2	1	1	2	0	2	2	2	2	2
0	0	1	1	0	1	0	0	1	2	2	1	1	1	1	1	1	1
0	0	0	1	2	2	0	1	2	2	1	0	1	1	1	1	1	2
0	0	1	0	2	2	0	0	2	1	2	2	1	1	1	1	2	2
0	0	2	2	0	1	0	0	1	2	2	2	1	1	2	2	1	1
0	0	2	1	0	2	0	0	2	2	1	2	1	1	1	2	2	2
0	0	0	2	1	1	0	2	2	1	2	0	1	1	2	1	2	2
0	0	2	0	1	1	0	0	2	2	2	2	1	1	2	2	1	2
0	0	1	1	0	2	0	2	2	2	2	0	1	1	2	2	2	2
0	0	1	2	0	1	0	1	1	1	1	1	1	2	2	2	2	1
0	0	0	2	2	2	0	2	1	1	1	1	1	2	2	2	2	2
0	0	2	0	2	2	0	1	1	1	1	2	2	2	2	2	2	2

In Table 2, 66 representatives of Ω for $l=3$ and $k=4$ are listed. For simplicity, the element ω_{pq} of $\boldsymbol{\omega}\in\Omega$ is written by pq in the first row of Table 2.

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