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(Citation)

Kobe Journal of Mathematics, 4(1):31-41

(Issue Date)

1987-06

(Resource Type)

journal article

(Version)

Version of Record

(JaLCD0I)

<https://doi.org/10.24546/E0034654>

(URL)

<https://hdl.handle.net/20.500.14094/E0034654>



ON THE INTEGRAL HOMOLOGY OF INFINITE CYCLIC COVERINGS OF LINKS

By Akio KAWAUCHI

(Received September 2, 1985)

§0. Introduction

We consider a polygonal oriented link ℓ in the Euclidean 3-space R^3 . Let $E(\ell) = R^3 - \ell$. We have a unique infinite cyclic covering space $\tilde{E}(\ell)$ over $E(\ell)$ associated with the epimorphism $\gamma: \pi_1(E(\ell)) \rightarrow \langle t \rangle$ sending each meridian element*) of ℓ to t , where $\langle t \rangle$ is the infinite cyclic group generated by a letter t . For the integral group ring $\Lambda = \mathbb{Z}\langle t \rangle$, the integral homology $H_1(\tilde{E}(\ell))$, which we denote by $H(\ell)$, forms a finitely generated Λ -module. Throughout this paper, Λ -modules will mean finitely generated Λ -modules, unless otherwise stated. For a Λ -module H , we use the following notations: DH = the (unique) maximal finite Λ -submodule of H (cf. [Ka]₄), $D_0H = \bigcap_{i=1}^{+\infty} (t-1)^i DH$, TH = the Λ -torsion part of H , $BH = H/TH$, $\beta H = \text{rank}_{\Lambda} H$, eH = the minimal number of elements generating H over Λ , and $E^q H = \text{Ext}_{\Lambda}^q(H, \Lambda)$. Note that $D_0H = (t-1)^N DH$ for a large positive integer N and hence $t-1: D_0H \cong D_0H$. By convention, $eH = 0$ if $H = 0$. It is known that $E^q H = 0$ for $q \geq 3$ and there are natural Λ -isomorphisms $E^2 H \cong E^2 DH \cong \text{Hom}_{\mathbb{Z}}(DH, \mathbb{Q}/\mathbb{Z})$ and H has the Λ -projective dimension ≤ 1 if and only if $DH = 0$ (cf. [Ka, §3]₄, Levine [L]). By the identity $\lambda(t) \cdot x = \lambda(t^{-1})x$ for $\lambda(t) \in \Lambda$ and $x \in H$, H has another Λ -module structure. We denote this Λ -module by $\bar{H}$. When $H = H(\ell)$, we denote DH , D_0H , TH , BH , βH and eH by $D(\ell)$, $D_0(\ell)$, $T(\ell)$, $B(\ell)$, $\beta(\ell)$ and $e(\ell)$, respectively. Let $\mu(\ell)$ be the number of components of ℓ . It is well-known that $\beta(\ell) \leq \mu(\ell) - 1$ and the equality holds for, e.g., a slice link ℓ in the strong sense (cf. [Ka]₁). Our first purpose is to observe that there are many links ℓ with $D(\ell) \neq 0$. Actually, we characterize $D_0(\ell)$ for all links ℓ .

THEOREM I. *For all links ℓ we have $eE^2 D_0(\ell) \leq \beta(\ell) \leq \mu(\ell) - 1$. Conversely, given a finite Λ -module D with $t-1: D \cong D$, then for any integers r, s with $eE^2 D \leq r \leq s$ we have a link ℓ such that $\mu(\ell) = s+1$, $\beta(\ell) = r$ and $D(\ell) = D_0(\ell) \cong D$.*

*) "a meridian element of ℓ " means "an element of $\pi_1(E(\ell))$ represented by a loop homotopic to a meridian of ℓ " (The orientation of a meridian of ℓ is uniquely specified by those of ℓ and R^3).

Remarks. (I.1) For all links ℓ we also have $eE^2D(\ell) \leq \beta(\ell)$ and $e(D(\ell)/(t-1)D(\ell)) \leq \mu(\ell) - 1 - \beta(\ell)$ (cf. Lemma 1.1). In particular, if $\beta(\ell) = 0$, then $D(\ell) = 0$. If $\beta(\ell) = \mu(\ell) - 1$, then $D(\ell) = D_0(\ell)$. So, by Theorem I, we can characterize $D(\ell)$ for all links ℓ with $\mu(\ell) \leq 2$. Hillman's link in [H, Fig. V.1], say ℓ_H , is a ribbon link in the strong sense with $\beta(\ell_H) = \mu(\ell_H) - 1 = 1$ and $D_0(\ell_H) = D(\ell_H) \cong \Lambda/(3, t-2)$. The link ℓ with $r=s$, constructed in the proof of Theorem I is also a ribbon link in the strong sense.

(I.2) A link ℓ is *reversible* if there is an auto-homeomorphism h of R^3 with $h(\ell) = \ell$ such that h is orientation-preserving and $h|_\ell$ is orientation-reversing or such that h is orientation-reversing and $h|_\ell$ is orientation-preserving. For a reversible link ℓ , there is a Λ -isomorphism $H(\ell) \cong \overline{H(\ell)}$. So, we obtain the following:

(I.2.1) If ℓ is reversible, then there is a Λ -isomorphism $D_0(\ell) \cong \overline{D_0(\ell)}$.

From this and Theorem I, we obtain many non-reversible links. For example, any links ℓ with $D_0(\ell) = \Lambda/(m, t-2)$ for any odd integer $m \geq 5$ are not reversible.

(I.3) If there is a disconnected Seifert surface for ℓ , then we say that ℓ is *weakly split* (cf. [Muras, Chap. V]). This is equivalent to saying that there is an epimorphism $\pi_1(E(\ell)) \rightarrow \langle t_1, t_2 \rangle$ sending each meridian element of ℓ to a conjugate of t_1 or t_2 (cf. Smythe [Sm]), where $\langle t_1, t_2 \rangle$ is the free group with basis t_1, t_2 . We say that ℓ is *homologically split* if there is an epimorphism $\pi_1(E(\ell)) \rightarrow \langle t_1, t_2 \rangle$ sending each meridian element of ℓ to an element congruent to t_1 or t_2 modulo the commutator subgroup $\langle t_1, t_2 \rangle'$. Clearly, a weakly split link is homologically split, but the converse is not true (cf. [Sm]). We note the following:

(I.3.1) If ℓ is homologically split, then $eE^2D_0(\ell) < \beta(\ell)$.

For example, Hillman's link ℓ_H has $eE^2D(\ell_H) = \beta(\ell_H) = 1$ and is not homologically split, shown by Hillman [H], using different methods. For each $r \geq 1$ there are many links ℓ with $eE^2D_0(\ell) = \beta(\ell) = r$, e.g., a ribbon link ℓ in the strong sense with $\beta(\ell) = \mu(\ell) - 1 = r$ and $D_0(\ell) = \bigoplus \Lambda/(m, t-2)$ for odd $m \geq 3$ (cf. (I.1)), which are not homologically split. To prove (I.3.1), let ℓ be homologically split. Then there is a Λ -epimorphism $H(\ell) \rightarrow \Lambda$. Hence $B(\ell) \cong B_1 \oplus \Lambda$ for a torsion-free Λ -module B_1 . By [Ka, §3]₄ there is a Λ -epimorphism $E^0E^0B_1 \rightarrow E^2E^1B_1$ and $E^0E^0B_1$ is Λ -free of rank $\beta B_1 = \beta(\ell) - 1$. $E^2E^1B_1 \cong E^2E^1B(\ell)$. The inclusion $D_0E^1B(\ell) \subset E^1B(\ell)$ induces a Λ -epimorphism $E^2E^1B(\ell) \rightarrow E^2D_0E^1B(\ell)$. We see from Lemma 1.1 that $\overline{D_0(\ell)} \cong D_0E^1B(\ell)$. So, $E^2D_0E^1B(\ell) \cong E^2\overline{D_0(\ell)}$. Then $eE^2D_0(\ell) = eE^2\overline{D_0(\ell)} = eE^2D_0E^1B(\ell) \leq eE^2E^1B(\ell) = eE^2E^1B_1 \leq eE^0E^0B_1 = \beta(\ell) - 1$, as desired.

Next, we study the invariant $e(\ell)$. If H has a square Λ -presentation matrix, then let mH be the minimal size of such a matrix. Otherwise, let $mH = +\infty$.

By convention, $mH=0$ if $H=0$. Clearly, $mH \geq eH$. A presentation for H is a Λ -epimorphism $\psi: P \rightarrow H$ with P , Λ -free of finite rank. We say that H is *normal* if $e(\ker \psi) \leq eP$ for any presentation $\psi: P \rightarrow H$. We have $mH = eH$ for all normal H . Let $m(\ell) = mH(\ell)$. Nakanishi [N]₁ introduced $m(\ell)$ when ℓ is a knot. For subsets $A \subset R^3$ and $J \subset R$, let $AJ = \{(x, t) \in R^4 \mid x \in A, t \in J\}$. Let $F \subset R^3[0, 2]$ be a compact oriented proper polyhedral locally flat surface. By [K/S/S, I], F is ambient isotopic to a surface with only elementary critical points (concerning the height function $R^3[0, 2] \rightarrow [0, 2]$ sending (x, t) to t). When F has only elementary critical points, let $c_0(F)$, $c_1(F)$ and $c_2(F)$ be the numbers of minimal, saddle and maximal points, respectively.

THEOREM II. (1) For all links ℓ , $H(\ell)$ is normal and in particular, $m(\ell) = e(\ell)$,
 (2) If two links $\ell_i \subset R^3[i]$, $i=0, 2$, bound a compact oriented proper surface $F \subset R^3[0, 2]$ with only elementary critical points, then we have $|c_2(F) + e(\ell_2) - c_0(F) - e(\ell_0)| \leq c_1(F)$.

REMARKS. (II.1) If H is of Λ -projective dimension ≤ 1 , then $\ker \psi$ of any presentation $\psi: P \rightarrow H$ is Λ -free by Seshadri [Se] and hence H is normal. However, this does not mean Theorem II (1), since there are many links ℓ with $E^2H(\ell) \neq 0$ by Theorem I.

(II.2) An analogue of Theorem II (1) to higher dimensional links does not hold in general. For example, a finite knot module $H(\neq 0)$, occurring as the first Alexander module of a high-dimensional knot must have $mH = +\infty$. In fact, by Crowell [C] and Levine [L], the following conditions on a knot module H are equivalent (cf. Weber [W]): (i) $mH < +\infty$, (ii) H is $\mathbb{Z}$ -torsion-free, (iii) H is of Λ -projective dimension ≤ 1 , (iv) $mH = eH$, (v) H is normal.

(II.3) Several estimates analogous to Theorem II (2) are known (cf. [H/K/N/S], [Ka]₃, [N]₁, [N]₂, [N/N] and [Sa]), but the present estimate combined with Theorem II (1) generalizes or improves them.

(II.4) Assume that a link ℓ' is obtained from a link ℓ by hyperbolic transformations along m bands (cf. [K/S/S, I]). Then we can construct a surface $F \subset R^3[0, 2]$ with no critical points except m saddle points such that $F \cap R^3[0] = \ell[0]$ and $F \cap R^3[2] = \ell'[2]$ (cf. [K/S/S, I]). By Theorem II (2), we obtain the following:

(II.4.1) If ℓ' is obtained from ℓ by hyperbolic transformations along m bands, then $|e(\ell') - e(\ell)| \leq m$.

For example, we have $e(k_B) \leq 4$ for all knots k_B obtained from the Borromean ring, say ℓ_B , by fusions, because $H(\ell_B) \cong \Lambda/(t-1)^2 \oplus \Lambda/(t-1)^2$. This gives a restriction to k_B . Note that the Robertello/Arf invariant, $R(k_B) \neq 0$ and the Alexander polynomials of k_B 's are not restrictive in the sense that they can have

all of the Alexander polynomials of knots k with $R(k) \neq 0$ (cf. Murakami [Murak]).

(II.5) Let ℓ^* be a link diagram on a plane R^2 representing a link ℓ . For an integer $s \neq 0$, a $T(s)$ -operation on ℓ^* is a replacement of a 2-string trivial braid in ℓ^* by a 2-string braid of s full twists (See Fig. 1). Let $u_w(\ell^*)$ be the minimal number of $T(s)$ -operations, where s is not fixed, on the link diagram ℓ^* needed to transform ℓ into a trivial link (of $\mu(\ell)$ components). The *weak unlinking number* $u_w(\ell)$ of ℓ is the minimal number of $u_w(\ell^*)$ taken over all link diagrams ℓ^* representing ℓ (cf. Kinoshita [Ki]). Restricting the $T(s)$ -operations to $T(\pm 1)$ -operations, we have the usual unlinking number $u(\ell)$ of ℓ . Clearly, $u(\ell) \geq u_w(\ell)$. The inequality $u(\ell) + \mu(\ell) - 1 \geq e(\ell)$ is well-known by a surgery description method (cf. Nakanishi [N]₁). We observe the following:

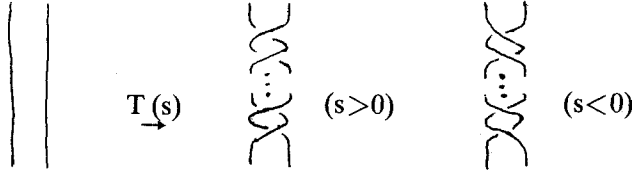


Fig. 1.

(II.5.1) For all links ℓ , $u_w(\ell) + \mu(\ell) - 1 \geq e(\ell)$.

To see this, let $u = u_w(\ell)$. Fig. 2 shows that there are u oriented links ℓ_i with $\mu(\ell_i) = 2$, $i = 1, 2, \dots, u$, such that the fusion of the split union $\ell^+ = \ell + \ell_1 + \dots + \ell_u$ along some $2u$ bands produce a trivial link 0 with $\mu(0) = \mu(\ell)$. By (II.4.1), $|e(\ell^+) - e(0)| \leq 2u$. Note that $H(\ell^+) \cong H(\ell) \oplus (\oplus_{i=1}^u H(\ell_i)) \oplus (\oplus_{i=1}^u \Lambda)$. Then $e(\ell^+) = e(H(\ell) \oplus (\oplus_{i=1}^u H(\ell_i))) + u \geq e(\ell) + u$ (cf. Lemma 2.5). Since $e(0) = \mu(\ell) - 1$, it follows that $e(\ell) + u - \mu(\ell) + 1 \leq 2u$, i.e., $e(\ell) \leq u_w(\ell) + \mu(\ell) - 1$.

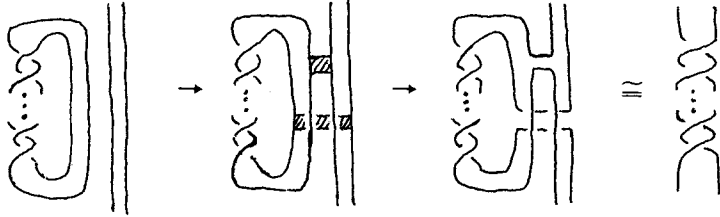


Fig. 2.

§ 1. Proof of Theorem I.

LEMMA 1.1. For all links ℓ we have the inequalities $eE^2D(\ell) \leq \beta(\ell)$, $e(D(\ell)/(t-1)D(\ell)) \leq \mu(\ell) - 1 - \beta(\ell)$ and $e(E^1B(\ell)/(t-1)E^1B(\ell)) \leq 2(\mu(\ell) - 1 - \beta(\ell))$ and a Λ -isomorphism $\overline{D_0}(\ell) \cong D_0E^1B(\ell)$.

If $\beta(\ell) = \mu(\ell) - 1$, then Lemma 1.1 means that $D(\ell) = D_0(\ell)$, $E^1B(\ell) = D_0E^1B(\ell)$ and $\overline{D(\ell)} \cong E^1B(\ell)$, for $E^1B(\ell)$ is finite (cf. [Ka, §3]₄).

1.2 PROOF OF LEMMA 1.1. Let $E^+ = E^+(\ell)$ be the compact manifold obtained from $S^3 = R^3 \cup \{\infty\}$ by removing an open tubular neighborhood of ℓ and $\tilde{E}^+ = \tilde{E}^+(\ell)$, an infinite cyclic covering space over E^+ determined by the covering space $\tilde{E}(\ell)$. Clearly, $H_1(\tilde{E}^+) \cong H(\ell)$. Let $H = H_1(\tilde{E}^+, \partial\tilde{E}^+)$. By the Second Duality Theorem of [Ka]₄, we have $\overline{D(\ell)} \cong \overline{DH_1(\tilde{E}^+)} \stackrel{\theta}{\cong} E^1BH$ and $\overline{DH} \stackrel{\theta}{\cong} E^1BH_1(\tilde{E}^+) \cong E^1B(\ell)$. By [Ka, §3]₄, there is a Λ -epimorphism $E^0E^0BH \rightarrow E^2E^1BH$ and E^0E^0BH is Λ -free of rank $\beta H = \beta(\ell)$. Since $E^2E^1BH \cong E^2\overline{D(\ell)}$, it follows that $eE^2D(\ell) = eE^2\overline{D(\ell)} = eE^2E^1BH \leq \beta(\ell)$. The Wang exact sequence (cf. [Mi]) $H \xrightarrow{t-1} H \rightarrow H_1(E^+, \partial E^+) \rightarrow 0$ induces an exact sequence $(S') \ 0 \rightarrow BH \xrightarrow{t-1} BH \rightarrow H' \rightarrow 0$, where H' is the quotient of $H_1(E^+, \partial E^+)$ by the image, T' , of TH . Note that $\text{rank}_Z H' = \beta(\ell)$, and $H_1(E^+, \partial E^+)$ and T' are Z -free of ranks $\mu(\ell) - 1$ and $\mu(\ell) - 1 - \beta(\ell)$, respectively. The sequence (S') induces an exact sequence $E^1BH \xrightarrow{t-1} E^1BH \rightarrow E^2H' \rightarrow 0$. So, $D(\ell)/(t-1)D(\ell) \cong E^1BH/(t-1)E^1BH \cong E^2H'$. The exact sequence $0 \rightarrow T' \rightarrow H_1(E^+, \partial E^+) \rightarrow H' \rightarrow 0$ induces an epimorphism $E^1T' \rightarrow E^2H'$. Since E^1T' is Z -free of rank $\mu(\ell) - 1 - \beta(\ell)$, we see that $e(D(\ell)/(t-1)D(\ell)) = eE^2H' \leq \mu(\ell) - 1 - \beta(\ell)$. The exact sequence $(S) \ H_1(\partial\tilde{E}^+) \xrightarrow{i_*} H_1(\tilde{E}^+) \xrightarrow{j_*} H \xrightarrow{\partial} \tilde{H}_0(\partial\tilde{E}^+) \rightarrow 0$ induces a short exact sequence $(S'') \ 0 \rightarrow B(\ell) \rightarrow BH \rightarrow H'' \rightarrow 0$, where H'' is the quotient of $\tilde{H}_0(\partial\tilde{E}^+)$ by $T'' = \partial TH$. Since the boundary map $H_1(E^+, \partial E^+) \rightarrow \tilde{H}_0(\partial\tilde{E}^+)$ sends T' onto the image of T'' under the isomorphism $\tilde{H}_0(\partial\tilde{E}^+) \cong \tilde{H}_0(\partial E^+)$, we have that $\text{rank}_Z T'' \leq \text{rank}_Z T' = \mu(\ell) - 1 - \beta(\ell)$. The sequence (S'') induces an exact sequence $E^1BH \rightarrow E^1B(\ell) \rightarrow E^2H'' \rightarrow 0$ and hence an exact sequence $E^1BH/(t-1)E^1BH \rightarrow E^1B(\ell)/(t-1)E^1B(\ell) \rightarrow E^2H'' \rightarrow 0$. Since $eE^2H'' \leq \mu(\ell) - 1 - \beta(\ell)$ [In fact, there is an epimorphism $E^1T'' \rightarrow E^2H''$ and E^1T'' is Z -free of rank $\leq \mu(\ell) - 1 - \beta(\ell)$] and $e(E^1BH/(t-1)E^1BH) = eE^2H' \leq \mu(\ell) - 1 - \beta(\ell)$, it follows that $e(E^1B(\ell)/(t-1)E^1B(\ell)) \leq 2(\mu(\ell) - 1 - \beta(\ell))$. Next, we show that the map $D_0j_*: D_0H_1(\tilde{E}^+) \rightarrow D_0H$ induced from j_* in the sequence (S) is a Λ -isomorphism. Clearly, D_0j_* is injective, for $(t-1)H_1(\partial\tilde{E}^+) = 0$ and $t-1: D_0H_1(\tilde{E}^+) \cong D_0H_1(\tilde{E}^+)$. Let $x \in D_0H$ and write $x = (t-1)x'$ for $x' \in D_0H$. There is a $y' \in H_1(\tilde{E}^+)$ so that $j_*(y') = x'$. For $y = (t-1)y'$, we have $j_*(y) = x$ and $y \in DH_1(\tilde{E}^+)$ [In fact, take positive integers m, m' so that $(t^m - 1)x' = m'x' = 0$. Then $(t^m - 1)y = m'y = 0$]. Choose a positive integer N so that $D_0H_1(\tilde{E}^+) = (t-1)^N \cdot DH_1(\tilde{E}^+)$ and $(t-1)^N$ acts trivially on D_0H . Then $(t-1)^Ny \in D_0H_1(\tilde{E}^+)$ and $D_0j_*((t-1)^Ny) = x$. Thus, D_0j_* is a Λ -isomorphism. Therefore, $\overline{D_0(\ell)} \cong \overline{D_0H_1(\tilde{E}^+)} \stackrel{\theta}{\cong} \overline{D_0H} \cong D_0E^1BH_1(\tilde{E}^+) \cong D_0E^1B(\ell)$. This completes the proof.

Let L be a polyhedral locally flat oriented 2-link (of 2-spheres) in R^4 , the number of whose components is denoted by $\mu(L)$. Let $E(L) = R^4 - L$ and $\tilde{E}(L)$, the infinite cyclic covering space over $E(L)$ associated with the epimorphism $\pi_1(E(L)) \rightarrow$

$\langle t_1, t_2 \rangle$ sending each meridian element of L to t . Let $H(L) = H_1(\tilde{E}(L))$, $TH = TH(L)$, $B(L) = BH(L)$ and $\beta(L) = \beta H(L)$.

LEMMA 1.3. *Assume that a link $\ell \subset R^3$ is the middle cross-sectional link of a 2-link $L \subset R^4$ in a normal form (cf. [K/S/S, I]). Then the inclusion $E(\ell) \subset E(L)$ induces a Λ -isomorphism $B(\ell) \cong B(L)$.*

PROOF. The natural map $H(\ell) \rightarrow H(L)$ is onto, for the natural map $\pi_1(E(\ell)) \rightarrow \pi_1(E(L))$ is onto by the van Kampen theorem. Since $H_2(E(L), E(\ell)) = 0$, we see from the Wang exact sequence (cf. [Mi]) that $BH_2(\tilde{E}(L), \tilde{E}(\ell)) = 0$. Thus, the exact sequence $H_2(\tilde{E}(L), \tilde{E}(\ell)) \rightarrow H(\ell) \rightarrow H(L) \rightarrow 0$ induces an isomorphism $B(\ell) \cong B(L)$, completing the proof.

The following characterizes $B(L)$ for all 2-links L :

LEMMA 1.4. *For a 2-link $L \subset R^4$, we have $\beta(L) = \mu(L) - 1$ and $t-1: E^1 B(L) \cong E^1 B(L)$. Conversely, given a finitely generated, torsion-free Λ -module B with $\beta B = r$ and $t-1: E^1 B \cong E^1 B$, then there is a 2-link $L \subset R^4$ with $\mu(L) = r+1$ and $H(L) = B(L) \cong B$.*

PROOF. Let ℓ be the middle cross-sectional link of L in a normal form. Then $\mu(\ell) = \mu(L)$. By Lemma 1.3, $B(\ell) \cong B(L)$ and $\beta(\ell) = \beta(L)$. Since ℓ is a ribbon link in the strong sense, we have $\beta(\ell) = \mu(\ell) - 1$. By Lemma 1.1, $t-1: E^1 B(\ell) \cong E^1 B(\ell)$. Hence $\beta(L) = \mu(L) - 1$ and $t-1: E^1 B(L) \cong E^1 B(L)$. Next, note that B is of Λ -projective dimension ≤ 1 (cf. [Ka, §3]₄). Let $0 \rightarrow P_1 \rightarrow P_0 \rightarrow B \rightarrow 0$ be a Λ -projective (i.e., Λ -free by [Se]) resolution for B with $eP_1 = d$, $eP_0 = d+r$ for some d . By [Ka, §3]₄, there is a Λ -exact sequence $0 \rightarrow B \rightarrow E^0 E^0 B \rightarrow E^2 E^1 B \rightarrow 0$. By assumption, $t-1: E^2 E^1 B \cong E^2 E^1 B$, so that $B/(t-1)B \cong E^0 E^0 B/(t-1)E^0 E^0 B$. Since $E^0 E^0 B$ is Λ -free of rank r , $B/(t-1)B$ is $\mathbb{Z}$ -free of rank r and the induced exact sequence $0 \rightarrow P_1/(t-1)P_1 \rightarrow P_0/(t-1)P_0 \rightarrow B/(t-1)B \rightarrow 0$ splits. Let $P(t) = (p_{ij}(t))$ be a $d \times (d+r)$ -matrix representing the Λ -monomorphism $P_1 \rightarrow P_0$. By elementary row and column operations on $P(t)$, we can assume that $p_{ij}(1) = \delta_{ij}$ for all i, j . Let $p_{ij}(t) = \sum_{n=-\infty}^{+\infty} a(i, j, n)t^n$, $a(i, j, n) \in \mathbb{Z}$. The construction of our 2-link is similar to the Kervaire/Levine construction of a 2-knot (cf. [Ke], [L]). Construct a group presentation $G = \langle y_1, \dots, y_{d+r}, z \mid R_1, \dots, R_d \rangle$ where

$$\begin{aligned} R_1 &= \prod_{n=-\infty}^{+\infty} (z^n y_1 z^{-n})^{a(1,1,n)} \dots \prod_{n=-\infty}^{+\infty} (z^n y_{d+r} z^{-n})^{a(1,d+r,n)} \\ &\dots \\ R_d &= \prod_{n=-\infty}^{+\infty} (z^n y_1 z^{-n})^{a(d,1,n)} \dots \prod_{n=-\infty}^{+\infty} (z^n y_{d+r} z^{-n})^{a(d,d+r,n)}. \end{aligned}$$

Let $\gamma: G \rightarrow \langle t \rangle$ be the epimorphism defined by $\gamma(y_j) = 1$ and $\gamma(z) = t$, and $\tilde{G}$, the kernel of γ . By the Fox free calculus, $\gamma(\partial R_i / \partial y_j) = p_{ij}(t)$ and $\gamma(\partial R_i / \partial z) = 0$, so that

$H_1(\tilde{G})$ has $P(t)$ as a Λ -presentation matrix (cf. [Ka]₂). Hence $H_1(\tilde{G}) \cong B$. Let $W = \#_{j=1}^d S^1 \times S_j^3 \# R^4$. Take a trivial 2-link L_0 with components S_i^2 , $i=0, 1, \dots, r$, in the factor R^4 of W . Identify $\pi_1(W - L_0)$ with the free group $\langle y_1, \dots, y_{d+r}, z \rangle$ so that y_j ($1 \leq j \leq d$), $y_{d+i}z$ ($1 \leq i \leq r$) and z are represented by loops homotopic to $S^1 \times *_j (\subset S^1 \times S_j^3)$, meridians of S_i^2 and S_0^2 , respectively. Let $C_1, \dots, C_d$ be mutually disjoint simple closed curves in $W - L_0$ such that C_i is homotopic to a loop representing the element $R_i \in \pi_1(W - L_0)$. Let $T_1, \dots, T_d$ be mutually disjoint tubular neighborhoods of $C_1, \dots, C_d$. Since $R_i(y_1, \dots, y_{d+r}, 1) = y_i$, $i=1, \dots, d$, the 4-manifold obtained from W by surgery along $T_1, \dots, T_d$ (with any framings) is R^4 . Let $L \subset R^4$ be a 2-link resulting from $L_0 \subset R^4$ by the surgery. Then $\pi_1(E(L)) \cong G$ and $H(L) \cong H_1(\tilde{G}) \cong B$. This completes the proof.

1.5 PROOF OF THEOREM I. The inclusion $D_0(\ell) \subset D(\ell)$ induces a Λ -epimorphism $E^2 D(\ell) \rightarrow E^2 D_0(\ell)$. So, by Lemma 1.1, $eE^2 D_0(\ell) \leq eE^2 D(\ell) \leq \beta(\ell) \leq \mu(\ell) - 1$. Let D be a finite Λ -module with $eE^2 D \leq r \leq s$ and $t-1: D \cong D$. First, let $r=s$. Since $eE^2 D \leq r$, there is a presentation $\psi: P \rightarrow E^2 D$ with $eP=r$. Let $B = \ker \psi$, which is Λ -torsion-free of rank r . Note that $E^1 B \cong E^2 E^2 D \cong D$. By Lemma 1.4, there is a 2-link $L \subset R^4$ with $\mu(L)=r+1$ and $H(L) \cong B$. Let $\ell \subset R^3$ be the middle cross-sectional link of L in a normal form. Then $\mu(\ell) = \mu(L) = r+1$ and ℓ is a ribbon link in the strong sense. By Lemma 1.3, $B(\ell) \cong B$. By Lemma 1.1, $\overline{D_0}(\ell) = \overline{D}(\ell) \cong E^1 B \cong D$. Let $\bar{\ell}$ be the link obtained from ℓ by reversing the orientation of ℓ . Then $D(\bar{\ell}) \cong \overline{D}(\ell) \cong D$. For $s=r+m$, $m \geq 1$, let ℓ_m be a link with $\mu(\ell_m) = m+1$, illustrated in Fig. 3.

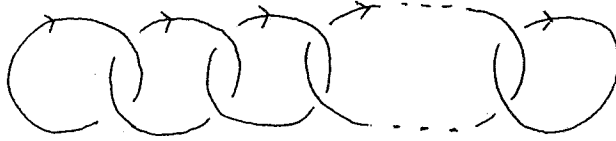


Fig. 3.

Then $H(\ell_m) \cong \oplus \Lambda/(t-1)$. Let $\bar{\ell}'$ be a link obtained from a split union $\bar{\ell} + \ell_m$ by a simple fusion so that $H(\bar{\ell}') \cong H(\bar{\ell}) \oplus H(\ell_m)$. Since $\mu(\bar{\ell}') = \mu(\bar{\ell}) + m = s+1$, $\beta(\bar{\ell}') = \beta(\bar{\ell}) = r$ and $D(\bar{\ell}') \cong D(\bar{\ell}) \cong D$, the link $\bar{\ell}'$ gives a desired one. This completes the proof.

REMARK 1.6. There is an argument on 2-links analogous to Remark (I.3). A 2-link L in R^4 is *weakly split* if there is a disconnected Seifert manifold for L in R^4 , which is equivalent that there is an epimorphism $\pi_1(E(L)) \rightarrow \langle t_1, t_2 \rangle$ sending each meridian element of L to a conjugate of t_1 or t_2 . A 2-link L in R^4 is *homologically split* if there is an epimorphism $\pi_1(E(L)) \rightarrow \langle t_1, t_2 \rangle$ sending each meridian element of L to an element congruent to t_1 or t_2 modulo $\langle t_1, t_2 \rangle'$.

(1.6.1) If L is homologically split, then $eE^2E^1B(L) < \beta(L)$.

The proof is basically contained in that of (I.3.1) and omitted. Let D be a finite A -module with $t-1: D \cong D$ and $eE^2D = r$. By an argument in 1.5, we have a 2-link L with $\beta(L) = \mu(L) - 1 = r$ and $E^1B(L) \cong D$. For example, taking $D = \bigoplus A/(m, t-2)(m, \text{odd} \geq 3)$, we see that for each $r \geq 1$ there are many 2-links L with $eE^2E^1B(L) = \beta(L) = \mu(L) - 1 = r$, that are not homologically split.

§2. Proof of Theorem II

Let W be a compact connected oriented 4-manifold with $H_1(W) = Z$. Let (A, A') be a splitting of ∂W , i.e., A, A' are compact 3-submanifolds of ∂W or empty such that $A' = \text{cl}_{\partial W}(\partial W - A)$. Let $\tilde{W}$ be the infinite cyclic connected covering space over W and $(\tilde{A}, \tilde{A}')$, the lift of (A, A') .

LEMMA 2.1. If $H_1(\tilde{W}) = 0$ and $\tilde{A}$ is connected, then $H_2(\tilde{W}, \tilde{A}')$ is A -free.

PROOF. $H_1(\tilde{W}, \tilde{A}) = 0$ and $H_0(\tilde{W}, \tilde{A}) \cong A/(t-1)$ (if $A = \emptyset$) or 0 (if $A \neq \emptyset$). By Reidemeister duality (cf. [Ka, §2]₄), there is a A -isomorphism $H_\lambda^q(\tilde{W}, \tilde{A}') \cong H_{4-q}(\tilde{W}, \tilde{A})$ for all q . By the universal coefficient exact sequence in [Ka, §1]₄, the sequence $0 \rightarrow BH_\lambda^q(\tilde{W}, \tilde{A}') \rightarrow E^0H_q(\tilde{W}, \tilde{A}') \rightarrow E^2H_{q-1}(\tilde{W}, \tilde{A}') \rightarrow TH_{\lambda}^{q+1}(\tilde{W}, \tilde{A}') \rightarrow E^1H_q(\tilde{W}, \tilde{A}') \rightarrow 0$ is A -exact for all q . We have $E^1H_2(\tilde{W}, \tilde{A}') = E^2H_2(\tilde{W}, \tilde{A}') = 0$. By [Ka, §3]₄, $H_2(\tilde{W}, \tilde{A}')$ is A -free, completing the proof.

Let G be a finitely generated group with an epimorphism $\gamma: G \rightarrow \langle t \rangle$ and $\tilde{G}$, the kernel of γ .

LEMMA 2.2. If G is isomorphic to the fundamental group of an orientable 3-manifold, then the A -module $H_1(\tilde{G})$ is normal.

PROOF. It is known that $H_1(\tilde{G}) = H_1(\tilde{M})$ for an infinite cyclic connected covering space $\tilde{M}$ over a compact orientable 3-manifold M such that ∂M is not empty and contains no 2-spheres (cf. [Ka]₂). For a presentation $\psi: P \rightarrow H_1(\tilde{M})$, let $g_1, \dots, g_r \in H_1(\tilde{M})$ be the image of a A -basis of P under ψ . Let $T_1, \dots, T_r$ be solid tori in $\text{Int } \tilde{M}$ whose cores represent $g_1, \dots, g_r$ and such that $t^n T_i \cap T_j = \emptyset$ except when $i = j$ and $n = 0$. The images $T'_1, \dots, T'_r$ of $T_1, \dots, T_r$ under the covering projection $\tilde{M} \rightarrow M$ are mutually disjoint solid tori in $\text{Int } M$. Construct a 4-manifold $W = M \times [0, 1] \cup D^2 \times D_1^2 \cup \dots \cup D^2 \times D_r^2$ identifying $T'_i \times 1$ (with any framing) with $(\partial D^2) \times D_i^2$, $i = 1, \dots, r$. There is an infinite cyclic connected covering $\tilde{W} \rightarrow W$ extending the covering $\tilde{M} \rightarrow M$ (identified with the covering $\tilde{M} \times 0 \rightarrow M \times 0$). By construction, $H_1(\tilde{W}) = 0$, so that $H_1(W) = Z$. Note that $\partial \tilde{W}$ is connected. By Lemma 2.1, $H_2(\tilde{W})$ is A -free. By assumption on M , the Euler characteristic, $\chi(M) \leq 0$. Since $H_q(W) \cong Z$ (if $q = 0, 1$), 0 (if $q = 3, 4$) and $\chi(W) =$

$\chi(M)+r$, it follows that $H_2(W)$ is $\mathbb{Z}$ -free of rank $\leq r$. Then we see from the Wang exact sequence that $eH_2(\tilde{W}) \leq r$. On the other hand, by excision, $H_2(\tilde{W}, \tilde{M}) \cong \bigoplus_{i=1}^r H_2(D^2 \times D_i^2, (\partial D^2) \times D_i^2) \otimes_{\mathbb{Z}} \Lambda \cong \bigoplus \Lambda$. Hence it follows from the exact sequence $H_2(\tilde{W}) \rightarrow H_2(\tilde{W}, \tilde{M}) \xrightarrow{\partial} H_1(\tilde{M}) \rightarrow 0$ that $H_1(\tilde{G}) = H_1(\tilde{M})$ is normal, since $\partial: H_2(\tilde{W}, \tilde{M}) \rightarrow H_1(\tilde{M})$ is a geometric realization of $\psi: P \rightarrow H_1(\tilde{M})$. This completes the proof.

2.3 PROOF OF THEOREM II (1). It is direct from Lemma 2.2.

REMARK 2.4. Let G be a group with an element of infinite order. Let $\tilde{G}_1 = \ker \gamma$ for a finitely generated subgroup G_1 of G with an epimorphism $\gamma: G_1 \rightarrow \langle t \rangle$. The Λ -module $H_1(\tilde{G}_1)$ is produced in G . By Lemma 2.2 and an argument of [Ka]₂, we have a necessary condition for G to be isomorphic to the fundamental group of a 3-manifold, which is different from that of [Ka]₂ and stated as follows:

(2.4.1) If G is isomorphic to the fundamental group of an orientable 3-manifold, then any Λ -module produced in G is normal.

(2.4.2) If G is isomorphic to the fundamental group of a non-orientable 3-manifold, then there exists an index 2 subgroup G' of G such that any Λ -module produced in G' is normal.

LEMMA 2.5. For a Λ -submodule H' of a finitely generated Λ -module H , let H/H' be of Λ -projective dimension ≤ 1 . Then

$$eH' + \beta(H/H') \leq eH \leq eH' + e(H/H').$$

PROOF. The right hand side inequality is obvious. Let $\psi: P \rightarrow H$ be a presentation with $eP = eH$. Let $P' = \psi^{-1}H'$. Then $P/P' \cong H/H'$. Since it is of Λ -projective dimension ≤ 1 , P' is Λ -projective, i.e., Λ -free by [Se] of rank $eH - \beta(H/H')$. Using that $\psi|P': P' \rightarrow H'$ is onto, we see that $eH' \leq eP' = eH - \beta(H/H')$. This completes the proof.

For example, if H/H' is Λ -free, then $\beta(H/H') = e(H/H')$ and hence $eH = eH' + e(H/H')$. The following special case of Theorem II (2) is obtained from a combination of Theorem II (1) and Nakanishi's inequality in [N]₂, but we shall give it a different proof.

LEMMA 2.6. When $c_2(F) = c_0(F) = 0$, we have $|e(\ell_2) - e(\ell_0)| \leq c_1(F)$.

PROOF. We may assume that $e(\ell_2) \geq e(\ell_0)$. Let $E^+(F)$ be the compact 4-manifold obtained from $S^3 \times [0, 2]$, $S^3 = R^3 \cup \{\infty\}$, by removing an open tubular neighborhood of F . For $i=0, 2$ let $E^+(\ell_i) = E^+(F) \cap S^3 \times i$ (cf. 1.2). Let W be a 4-manifold constructed in the proof of Lemma 2.2 where we take

$M = E^+(\ell_0)$, $\tilde{M} = \tilde{E}^+(\ell_0)$ and for $\psi: P \rightarrow H_1(\tilde{M})$, $eP = eH_1(\tilde{M}) = e(\ell_0)$. Construct a 4-manifold $W' = W \cup E^+(F)$ identifying two copies of $E^+(\ell_0)$. Since the natural map $\pi_1(E^+(\ell_0)) \rightarrow \pi_1(E^+(F))$ is onto by the van Kampen theorem, it follows that $H_1(W') = Z$ and $H_1(\tilde{W}') = 0$ for the connected infinite cyclic covering space $\tilde{W}'$. Note that the covering $\tilde{W}' \rightarrow W'$ is an extension of the coverings $\tilde{E}^+(\ell_i) \rightarrow E^+(\ell_i)$, $i=0, 2$. By Lemma 2.1, $H_2(\tilde{W}', \tilde{E}^+(\ell_2))$ is A -free. Since $H_2(E^+(F), E^+(\ell_2)) \rightarrow H_2(W', E^+(\ell_2)) \rightarrow H_2(W', E^+(F))$ is exact and $H_2(E^+(F), E^+(\ell_2)) \cong \bigoplus_{c_1(F)} Z$ and $H_2(W', E^+(F)) \cong H_2(W, E^+(\ell_0)) \cong \bigoplus_{e(\ell_0)} Z$, we see from the Wang exact sequence that $eH_2(\tilde{W}', \tilde{E}^+(\ell_2)) = \text{rank}_Z H_2(\tilde{W}', E^+(\ell_2)) \leq c_1(F) + e(\ell_0)$. But, the boundary map $\partial: H_2(\tilde{W}', \tilde{E}^+(\ell_2)) \rightarrow H_1(\tilde{E}^+(\ell_2))$ is onto. Hence $e(\ell_2) \leq c_1(F) + e(\ell_0)$. This completes the proof.

2.7 PROOF OF THEOREM II (2). F can be deformed into a surface F' in $R^3[0, 2]$ with only elementary critical points such that for each i , $i=0, 2$, $F' \cap R^3[i] = \ell'_i$ is a split union of ℓ_i and a trivial link of $c_i(F)$ components and $c_1(F') = c_1(F)$ and $c_i(F') = 0$ (cf. [K/S/S, I, §3]). By Lemma 2.6, $|e(\ell'_2) - e(\ell'_0)| \leq c_1(F)$. $H(\ell'_i) \cong H(\ell_i) \oplus (\bigoplus_{c_i(F)} A)$ and by Lemma 2.5, $e(\ell'_i) = e(\ell_i) + c_i(F)$, $i=0, 2$. Hence $|e(\ell_2) + c_2(F) - e(\ell_0) - c_0(F)| \leq c_1(F)$. This completes the proof.

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