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A CLASS OF MEROMORPHIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS

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ABSTRACT

In this paper, we introduce a class $\Sigma_p(\alpha, \beta, \gamma)$ of meromorphic univalent function in $D = \{z: 0 < |z| < 1\}$ and investigate coefficient estimates, distortion properties, radii of starlikeness and convexity for this class. Furthermore, it is shown that the class $\Sigma_p(\alpha, \beta, \gamma)$ is closed under convex linear combinations and convolutions.

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1. Introduction. Let Σ denote the class of functions of the form

$$(1.1) \quad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$$

which are regular in $D = \{z: 0 < |z| < 1\}$ with a simple pole at the origin with residue 1 there. And let Σ_s denote the subclass of Σ consisting of analytic and univalent functions $f(z)$ in D . A function $f(z)$ in Σ_s is said to be meromorphically starlike of order α if

$$(1.2) \quad \operatorname{Re} \left\{ - \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in D)$$

for some $\alpha (0 \leq \alpha < 1)$. We denote $\Sigma^*(\alpha)$ the class of all meromorphically starlike functions of order α . Similarly, a function $f(z)$ in Σ_s is said to be meromorphically convex order α if

$$(1.3) \quad \operatorname{Re} \left\{ - \left(1 + \frac{zf''(z)}{f'(z)} \right) \right\} > \alpha, \quad (z \in D)$$

for some $\alpha (0 \leq \alpha < 1)$. And we denote by $\Sigma_k(\alpha)$ the class of all meromorphically convex functions of order α . The class $\Sigma^*(\alpha)$ and similar classes have been extensively studied by Pommerenke [7], Clunie [1], Miller [3] and others.

Let Σ_p denote the class of functions of the form

$$(1.4) \quad f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \quad (a_n \geq 0)$$

that are analytic and univalent in D . Recently, Mogra, Reddy and Juneja [4] defined the class $\Sigma_p^*(\alpha, \beta)$ of meromorphically starlike functions of order α and type β which is a subclass of Σ_p and investigated interesting properties for the class $\Sigma_p^*(\alpha, \beta)$.

2. Coefficient estimates. We begin with the definition of $\Sigma(\alpha, \beta, \gamma)$.

DEFINITION 2.1. A function $f(z)$ in Σ is in the class $\Sigma(\alpha, \beta, \gamma)$ if it satisfies the condition

$$|z^2 f'(z) + 1| < \beta |(2\gamma - 1)z^2 f'(z) + (2\alpha\gamma - 1)| \quad (z \in D)$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$ and $\gamma \left(\frac{1}{2} \leq \gamma \leq 1\right)$.

THEOREM 2.2. Let $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ be analytic in D . If

$$(2.1) \quad \sum_{n=1}^{\infty} n(1 + 2\beta\gamma - \beta) |a_n| \leq 2\beta\gamma(1 - \alpha),$$

for some $\alpha (0 \leq \alpha < 1)$, $\beta (0 < \beta \leq 1)$ and $\gamma \left(\frac{1}{2} \leq \gamma \leq 1\right)$, then $f(z)$ is in the class $\Sigma(\alpha, \beta, \gamma)$.

PROOF. Suppose (2.1) holds for all admissible values of α, β, γ . Then we have

$$\begin{aligned} & |z^2 f'(z) + 1| - \beta |z^2 f'(z)(2\gamma - 1) + (2\alpha\gamma - 1)| \\ &= \left| \sum_{n=1}^{\infty} n a_n z^{n+1} \right| - \beta |2\gamma(\alpha - 1) + \sum_{n=1}^{\infty} (2\gamma - 1) n a_n z^{n+1}| \\ &\leq \sum_{n=1}^{\infty} n(1 + 2\beta\gamma - \beta) |a_n| |z|^{n+1} - 2\beta\gamma(1 - \alpha). \end{aligned}$$

Since the above inequality holds for all $r = |z|$, $0 < r < 1$, letting $r \rightarrow 1$, we have

$$\sum_{n=1}^{\infty} n(1 + 2\beta\gamma - \beta) |a_n| \leq 2\beta\gamma(1 - \alpha)$$

by (2.1). Hence it follows that $f(z)$ is in the class $\Sigma(\alpha, \beta, \gamma)$.

Let us write $\Sigma_p(\alpha, \beta, \gamma) = \Sigma_p \cap \Sigma(\alpha, \beta, \gamma)$ where Σ_p is the class of functions of the form (1.4) that are analytic and univalent in D .

THEOREM 2.3. Let the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ ($a_n \geq 0$) be analytic in D . Then $f(z)$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$ if and only if (2.1) is satisfied.

PROOF. By Theorem 2.2, it is sufficient to show the 'only if' part. Let us assume that $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ ($a_n \geq 0$) is in the class $\Sigma_p(\alpha, \beta, \gamma)$. Then

$$\begin{aligned} & \left| \frac{z^2 f'(z) + 1}{(2\gamma - 1)z^2 f'(z) + (2\alpha\gamma - 1)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} n a_n z^{n+1}}{(2\gamma - 1)(1 - \sum_{n=1}^{\infty} n a_n z^{n+1}) - (2\alpha\gamma - 1)} \right| < \beta, \quad (z \in U). \end{aligned}$$

Using the fact that $\operatorname{Re}(z) \leq |z|$ for all z ,

$$(2.2) \quad \operatorname{Re} \left\{ \frac{\sum_{n=1}^{\infty} n a_n z^{n+1}}{(2\gamma - 1)(1 - \sum_{n=1}^{\infty} n a_n z^{n+1}) - (2\alpha\gamma - 1)} \right\} < \beta, \quad (z \in U).$$

Now choose the values of z on real axis so that $f'(z)$ is real. Upon clearing the denominator in (2.2) and letting $z \rightarrow 1$ through positive values, we obtain

$$\sum_{n=1}^{\infty} n(1 + 2\beta\gamma - \beta)a_n \leq 2\beta\gamma(1 - \alpha).$$

Hence the result follows.

COROLLARY 2.4. If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$, then

$$(2.3) \quad a_n \leq \frac{2\beta\gamma(1 - \alpha)}{n(1 + 2\beta\gamma - \beta)}, \quad (n \geq 1).$$

The result is sharp for the function

$$(2.4) \quad f_n(z) = \frac{1}{z} + \frac{2\beta\gamma(1 - \alpha)}{n(1 + 2\beta\gamma - \beta)} z^n, \quad (n \geq 1).$$

3. Distortion properties and radii of starlikeness and convexity.

THEOREM 3.1. If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$, then

$$\frac{1}{|z|} - \frac{2\beta\gamma(1 - \alpha)}{(1 + 2\beta\gamma - \beta)} |z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{2\beta\gamma(1 - \alpha)}{(1 + 2\beta\gamma - \beta)} |z|.$$

The result is sharp.

PROOF. Suppose $f(z)$ is in $\Sigma_p(\alpha, \beta, \gamma)$. By Theorem 2.3, we have

$$\sum_{n=1}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)}.$$

Thus

$$\begin{aligned} |f(z)| &\leq \frac{1}{|z|} + |z| \sum_{n=1}^{\infty} a_n \\ &\leq \frac{1}{|z|} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} |z|. \end{aligned}$$

Also,

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - |z| \sum_{n=1}^{\infty} a_n \\ &\geq \frac{1}{|z|} - \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} |z|. \end{aligned}$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{(1+2\beta\gamma-\beta)} z.$$

THEOREM 3.2. If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$, then $f(z)$ is meromorphically starlike of order δ ($0 \leq \delta < 1$) in $|z| < r = r(\alpha, \beta, \gamma, \delta)$, where

$$r(\alpha, \beta, \gamma, \delta) = \inf_{n \geq 1} \left\{ \frac{n(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{\frac{1}{n+1}}.$$

The result is sharp.

PROOF. Let $f(z)$ be in the class $\Sigma_p(\alpha, \beta, \gamma)$. Then, by Theorem 2.3,

$$(3.1) \quad \sum_{n=1}^{\infty} \frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_n \leq 1.$$

It is sufficient to show that

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| \leq 1 - \delta$$

for $|z| < r(\alpha, \beta, \gamma, \delta)$, where $r(\alpha, \beta, \gamma, \delta)$ is as specified in the statement of the theorem. Then

$$\left| 1 + \frac{zf'(z)}{f(z)} \right| = \left| \frac{\sum_{n=1}^{\infty} (n+1)a_n z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n} \right|$$

$$\leq \frac{\sum_{n=1}^{\infty} (n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n |z|^{n+1}}.$$

This will be bounded by $1 - \delta$ if

$$(3.2) \quad \sum_{n=1}^{\infty} \frac{(n+2-\delta)}{1-\delta} a_n |z|^{n+1} \leq 1.$$

By (3.1), it follows that (3.2) is true if

$$\frac{(n+2-\delta)}{1-\delta} |z|^{n+1} \leq \frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)}, \quad (n \geq 1)$$

or

$$(3.3) \quad |z| \leq \left\{ \frac{n(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma(1-\alpha)(n+2-\delta)} \right\}^{\frac{1}{n+1}}, \quad (n \geq 1).$$

Setting $|z| = r(\alpha, \beta, \gamma, \delta)$ in (3.3), the result follows. The result is sharp for the functions

$$(3.4) \quad f_n(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} z^n, \quad (n \geq 1).$$

Using the method of Theorem 3.2, we obtain the following.

THEOREM 3.3. *If the function $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$, then $f(z)$ is meromorphically convex of order δ ($0 \leq \delta < 1$) in $|z| < r = r(\alpha, \beta, \gamma, \delta)$, where*

$$r(\alpha, \beta, \gamma, \delta) = \inf_{n \geq 1} \left\{ \frac{(1+2\beta\gamma-\beta)(1-\delta)}{2\beta\gamma((1-\alpha)(n+2-\delta))} \right\}^{\frac{1}{n+1}}.$$

The result is sharp for the functions defined by (3.4).

4. Convex linear combinations and convolution properties.

THEOREM 4.1. *Let $f_0(z) = \frac{1}{z}$*

and

$$f_n(z) = \frac{1}{z} + \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} z^n, \quad (n \geq 1).$$

Then the function $f(z)$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z),$$

where $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$.

PROOF. Let $f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z)$ with $\lambda_n \geq 0$ and $\sum_{n=0}^{\infty} \lambda_n = 1$. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \lambda_n f_n(z) \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \lambda_n \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} z^n. \end{aligned}$$

Since

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} \lambda_n \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)} \\ = \sum_{n=1}^{\infty} \lambda_n = 1 - \lambda_0 \leq 1, \end{aligned}$$

by Theorem 2.3, $f(z)$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$.

Conversely, suppose the function $f(z)$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$. Since

$$a_n \leq \frac{2\beta\gamma(1-\alpha)}{n(1+2\beta\gamma-\beta)}, \quad (n \geq 1),$$

setting

$$\lambda_n = \frac{n(1+2\beta\gamma-\beta)}{2\beta\gamma(1-\alpha)} a_n, \quad (n \geq 1)$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n,$$

it follows that

$$f(z) = \sum_{n=0}^{\infty} \lambda_n f_n(z).$$

This completes the proof of the theorem.

THEOREM 4.2. *The class $\Sigma_p(\alpha, \beta, \gamma)$ is closed under convex linear combinations.*

PROOF. Suppose $f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n$, $(i=1, 2)$ are in the class $\Sigma_p(\alpha, \beta, \gamma)$.
Let

$$f(z) = (1-t)f_1(z) + tf_2(z), \quad (0 \leq t \leq 1).$$

Then

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} ((1-t)a_{n,1} + ta_{n,2})z^n.$$

By Theorem 2.3, we have

$$\begin{aligned} \sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)((1-t)a_{n,1} + ta_{n,2}) \\ \leq (1-t)2\beta\gamma(1-\alpha) + t2\beta\gamma(1-\alpha) = 2\beta\gamma(1-\alpha). \end{aligned}$$

Therefore it follows that $f(z)$ is in the class $\Sigma_p(\alpha, \beta, \gamma)$.

Robertson [8] has shown that if $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n$ are in Σ_s , then so is their convolution $f * g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n$.

THEOREM 4.3. *If the functions $f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n$ are in the class $\Sigma_p(\alpha_i, \beta, \gamma)$ for each $i=1, 2, \dots, m$, respectively, then the convolution $f_1 * \dots * f_m(z)$ is in the class $\Sigma_p(1 - \prod_{i=1}^m (1-\alpha_i), \beta, \gamma)$.*

PROOF. Since the functions $f_i(z)$ are in the class $\Sigma_p(\alpha_i, \beta, \gamma)$ for each $i=1, 2, \dots, m$, respectively, by Theorem 2.3,

$$\sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta)a_{n,i} \leq 2\beta\gamma(1-\alpha_i)$$

and

$$a_{n,i} \leq \frac{2\beta\gamma(1-\alpha_i)}{1+2\beta\gamma-\beta}$$

for any $n \geq 1$. Hence we have

$$\begin{aligned} \sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta) \prod_{i=1}^m a_{n,i} &= \sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta) \prod_{i=1}^{m-1} \frac{2\beta\gamma(1-\alpha_i)}{(1+2\beta\gamma-\beta)} a_{n,m} \\ &= \frac{(2\beta\gamma)^{m-1}}{(1+2\beta\gamma-\beta)^{m-1}} \prod_{i=1}^{m-1} (1-\alpha_i) \sum_{n=1}^{\infty} n(1+2\beta\gamma-\beta) a_{n,m} \\ &\leq \frac{(2\beta\gamma)^m}{(1+2\beta\gamma-\beta)^{m-1}} \prod_{i=1}^m (1-\alpha_i) \end{aligned}$$

$$\leq 2\beta\gamma(1 - (1 - \prod_{n=1}^m (1 - \alpha_i)))\}.$$

Hence the result follows.

COROLLARY 4.4. *The class $\Sigma_p(\alpha, \beta, \gamma)$ is closed under the convolution.*

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