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LINKING 2-SPHERES IN THE 4-SPHERE

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In this paper, we shall show that every (piecewise-linear or smooth) link of two 2-spheres $S^2 \cup S^2$ in the 4-sphere S^4 can be separated by a homotopy which keeps the components disjoint. This result is a solution to the problem proposed by Dale Rolfsen at the Canadian Math. Soc. Annual Meeting, June 1983.

Throughout the paper, we work in the piecewise-linear category, consisting of simplicial complexes and piecewise-linear maps.

1. Definitions and our main theorem

An n -link with c components is an embedding $L: \sum_{i=1}^c S_i^n \rightarrow S^{n+2}$ (or R^{n+2}) of a disjoint union $\sum_{i=1}^c S_i^n$ of oriented n -spheres into the oriented $(n+2)$ -sphere S^{n+2} (or $(n+2)$ -space R^{n+2}). In particular, an n -link with one component will be called an n -knot.

Two n -links $L_j: \sum_{i=1}^c S_i^n \rightarrow S^{n+2}$ with c components, $j=1, 2$, are said to be *equivalent* (or *of the same type*), iff there exists an orientation preserving homeomorphism $\psi: S^{n+2} \rightarrow S^{n+2}$ such that $\psi L_1 = L_2$. An n -link $L: \sum_{i=1}^c S_i^n \rightarrow S^{n+2}$ is called *trivial* (or *unknotted*), iff there exists an embedding $\tilde{L}: \sum_{i=1}^c D_i^{n+1} \rightarrow S^{n+2}$ of a disjoint union $\sum_{i=1}^c D_i^{n+1}$ of oriented $(n+1)$ -disks with $\tilde{L}|_{\sum_{i=1}^c \partial D_i^{n+1}} = L$. Any two trivial n -links with c components are equivalent, and the equivalence class will be called the *trivial type*.

An n -link $L: \sum_{i=1}^c S_i^n \rightarrow S^{n+2}$ with $c \geq 2$ components is said to be *splittable*, iff there exist mutually disjoint $(n+2)$ -disks $D_1^{n+2}, \dots, D_c^{n+2}$ in S^{n+2} such that $L(S_i^n) \subset {}^\circ D_i^{n+2}$ for $i=1, \dots, c$. Here ${}^\circ X$ stands for the interior of a manifold X . A trivial n -link is, of course, splittable.

Examples of non-trivial and/or non-splittable 2-links can be seen in, for example, Artin [2], van Kampen [16], Andrews-Curtis [1], Kinoshita [9], Epstein [3], Yanagawa [18], Zeeman [19], etc., see also Suzuki [15]. As an n -link $L: \sum_{i=1}^c S_i^n \rightarrow S^{n+2}$, we refer to the image $(L(\sum_{i=1}^c S_i^n) \subset S^{n+2}) = (K_1 \cup \dots \cup K_c \subset S^{n+2})$, where $L(S_i^n) = K_i$ ($i=1, \dots, c$), and we shall use the motion picture method to

describe the configuration of subspaces of R^4 , $R^4 \cup \{\infty\} = S^4$, and we use the same notation and definitions as Kawauchi-Shibuya-Suzuki [7] and Suzuki [15]. Then we have the following:

1.1. PROPOSITION. (Cf. [7, Th. 2.1 and Cor. 2.16], [15, Th. 2.4 and 2.6]). *Any locally flat 2-link $(K_1 \cup \cdots \cup K_c \subset R^4)$ can be deformed into a 2-link $(\tilde{K}_1 \cup \cdots \cup \tilde{K}_c \subset R^4)$ in the normal form by an ambient isotopy of R^4 ; that is, $(\tilde{K}_1 \cup \cdots \cup \tilde{K}_c \subset R^4)$ is in the following position:*

- (1) *all maximum-disks of $\tilde{K}_1 \cup \cdots \cup \tilde{K}_c$ are in the hyperplane $R^3[3]$,*
- (2) *all minimum-disks of $\tilde{K}_1 \cup \cdots \cup \tilde{K}_c$ are in the hyperplane $R^3[-3]$,*
- (3) *all saddle-bands of $\tilde{K}_1 \cup \cdots \cup \tilde{K}_c$ are in the hyperplanes $R^3[-1]$ and $R^3[1]$, and*
- (4) *the equatorial cross-sectional 1-link $((\tilde{K}_1 \cup \cdots \cup \tilde{K}_c) \cap R^3[0] \subset R^3[0])$ is a 1-link with c components; and so the equatorial cross-section $(\tilde{K}_i \cap R^3[0] \subset R^3[0])$ is a 1-knot ($i=1, \dots, c$).*

The following unknotting criterion of 2-knots was first written down by the first author of the paper in [6], though its proof of [6, Lemma 2] was incomplete, latter to be filled in by Scharlemann [13, Main Theorem]. (Refer the reader to [15, §10] and [8, §5] for related topics.)

1.2. THEOREM. (Hosokawa [6], Scharlemann [13]). *Let $(K^2 \subset R^4)$ be a locally flat 2-knot in the normal form such that $K^2 \subset R^3[-3, 3]$ is the closed realizing 2-sphere of a sequence $\mathcal{O}_\mu \rightarrow \mathcal{X} \rightarrow \tilde{\mathcal{O}}_\alpha$, where $\mathcal{O}_\mu$ and $\tilde{\mathcal{O}}_\alpha$ are trivial 1-links with μ and α components, respectively, and $\mathcal{X}$ is a 1-knot. Suppose that $\mathcal{X}$ is obtained from $\mathcal{O}_\mu$ by $(\mu-1)$ -fusion along bands $B_1, \dots, B_{\mu-1}$ and $\tilde{\mathcal{O}}_\alpha$ is obtained from $\mathcal{X}$ by $(\alpha-1)$ -fission along bands $\tilde{B}_1, \dots, \tilde{B}_{\alpha-1}$; $\mathcal{X} = h(\mathcal{O}_\mu; B_1, \dots, B_{\mu-1})$, $\tilde{\mathcal{O}}_\alpha = h(\mathcal{X}; \tilde{B}_1, \dots, \tilde{B}_{\alpha-1})$.*

If we can rearrange $B_1, \dots, B_{\mu-1}$, $\tilde{B}_1, \dots, \tilde{B}_{\alpha-1}$, if necessary, so that in the sequence of 1-links

$$\begin{aligned} \mathcal{O}_\mu &\longrightarrow L_1 = h(\mathcal{O}_\mu; B_1) \longrightarrow L_2 = h(L_1; B_2) \\ &\longrightarrow \cdots \longrightarrow L_{\mu-1} = h(L_{\mu-2}; B_{\mu-1}) = \mathcal{X} \\ &\longrightarrow \tilde{L}_1 = h(\mathcal{X}; \tilde{B}_1) \longrightarrow \tilde{L}_2 = h(\tilde{L}_1; \tilde{B}_2) \\ &\longrightarrow \cdots \longrightarrow \tilde{L}_{\alpha-1} = h(\tilde{L}_{\alpha-2}; \tilde{B}_{\alpha-1}) = \tilde{\mathcal{O}}_\alpha, \end{aligned}$$

all 1-links L_i ($i=1, \dots, \mu-1$) and $\tilde{L}_j$ ($j=1, \dots, \alpha-1$) are trivial, then $(K^2 \subset R^4)$ is trivial.

1.3. DEFINITION. (1) Let $P_1, \dots, P_c$ be polyhedra (possibly not connected) and let $P = \sum_{i=1}^c P_i$ be the disjoint union, and let X be a topological space. Let

f_0 and f_1 be continuous maps of P into X such that $f_j(P_1), \dots, f_j(P_c)$ are mutually disjoint for both $j=0, 1$. Then f_0 and f_1 will be called *link-homotopic*, denoted by $f_0 \simeq_l f_1$, iff there exists a homotopy $\{\eta_t\}_{t \in I}: P \rightarrow X$ such that $\eta_0 = f_0$, $\eta_1 = f_1$ and the sets $\eta_t(P_1), \dots, \eta_t(P_c)$ are mutually disjoint for each value $t \in I = [0, 1]$.

This type of homotopy was studied by Milnor [10] for 1-links. We call such a homotopy $\{\eta_t\}_{t \in I}$ a *link-homotopy* between f_0 and f_1 .

(2) We suppose that there exist a polyhedron P and two continuous maps f_0 and f_1 of P into X such that $f_0(P) = Q_0$, $f_1(P) = Q_1$. If f_0 and f_1 is link-homotopic (by a link-homotopy $\{\eta_t\}_{t \in I}$), we will say that the subspaces Q_0 and Q_1 of X are *link-homotopic* (by a link-homotopy $\{\eta_t\}_{t \in I}$). In the following, we do not explain such a polyhedron P and maps f_0 and f_1 explicitly, unless there is danger of confusion.

We can now formulate our main theorem, to fix ideas; but the proof will require some lemmas before we start it in §3.

1.4. THEOREM. Every 2-link $L: S_1^2 \cup S_2^2 \rightarrow S^4$ (or R^4) is link-homotopic to a trivial 2-link.

From our definitions, as an immediate consequence of this theorem, we have the result stated in the Introduction to this paper.

We would like to thank Professor Dale Rolfsen for useful informations on this subject.

2. Some technical lemmas

2.1. LEMMA (Goldsmith [5]). Let $D_0, D_1, \dots, D_\mu, D'_0, D'_1, \dots, D'_\lambda$ be mutually disjoint disks on the hyperplane R^2 in the 3-space R^3 , which are oriented coherently with that of R^2 , and let $\partial D_i = O_i$ ($i=0, 1, \dots, \mu$), $\partial D'_j = O'_j$ ($j=0, 1, \dots, \lambda$), $\mathcal{O} = O_0 \cup O_1 \cup \dots \cup O_\mu$, $\mathcal{O}' = O'_0 \cup O'_1 \cup \dots \cup O'_\lambda$.

Let $B_1, \dots, B_\mu, B'_1, \dots, B'_\lambda$ be mutually disjoint oriented bands in R^3 such that each band B_i spans O_i and O_0 ($i=1, \dots, \mu$) and each bands B'_j spans O'_j and O'_0 ($j=1, \dots, \lambda$); that is, $h(\mathcal{O}; B_1, \dots, B_\mu)$ and $h(\mathcal{O}'; B'_1, \dots, B'_\lambda)$ are 1-knots (see [7, Def. 1.1]). Let $B_1^*, \dots, B_\mu^*, B_1'^*, \dots, B_\lambda'^*$ be mutually disjoint oriented bands in R^2 such that B_i^* spans O_i and O_0 with $B_i^* \cap O_i = B_i \cap O_i$, $B_i^* \cap O_0 = B_i \cap O_0$ ($i=1, \dots, \mu$), and $B_j'^*$ spans O'_j and O'_0 with $B_j'^* \cap O'_j = B'_j \cap O'_j$, $B_j'^* \cap O'_0 = B'_j \cap O'_0$ ($j=1, \dots, \lambda$).

Then, there exists a link-homotopy $\{\eta_t\}_{t \in I}$ between the polyhedra $(\mathcal{O} \cup B_1 \cup \dots \cup B_\mu) \cup (\mathcal{O}' \cup B'_1 \cup \dots \cup B'_\lambda)$ and $(\mathcal{O} \cup B_1^* \cup \dots \cup B_\mu^*) \cup (\mathcal{O}' \cup B_1'^* \cup \dots \cup B_\lambda'^*)$ in R^3 such that

(i) $\eta_t(O_0 \cup O'_0) = O_0 \cup O'_0$ for any $t \in I$, and so $\eta_1(O_i) = O_i$, $\eta_1(B_i) = B_i^*$ ($i=1, \dots, \mu$), $\eta_1(O'_j) = O'_j$, $\eta_1(B'_j) = B_j'^*$ ($j=1, \dots, \lambda$) and

(ii) $\eta_t(O_i) \cap \eta_t(O_k) = \emptyset$ ($i \neq k$) and $\eta_t(O'_j) \cap \eta_t(O'_h) = \emptyset$ ($j \neq h$) for any $t \in I$.

Proof. This lemma is essentially the same as Lemma 2.3 of Goldsmith [5]. In fact, the 1-link $h(\emptyset; B_1, \dots, B_\mu) \cup h(\emptyset'; B'_1, \dots, B'_\lambda)$ is a ribbon 1-link with 2 components. It should be noted that if $(B_1 \cup \dots \cup B_\mu \cup B'_1 \cup \dots \cup B'_\lambda) \cap ({}^\circ D_0 \cup {}^\circ D_1 \cup \dots \cup {}^\circ D_\mu \cup {}^\circ D'_0 \cup {}^\circ D'_1 \cup \dots \cup {}^\circ D'_\lambda) = \emptyset$, then $(D_0 \cup D_1 \cup \dots \cup D_\mu \cup B_1 \cup \dots \cup B_\mu) \cup (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda \cup B'_1 \cup \dots \cup B'_\lambda)$ is ambient isotopic to $(D_0 \cup D_1 \cup \dots \cup D_\mu \cup B_1^* \cup \dots \cup B_\mu^*) \cup (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda \cup B'_1{}^* \cup \dots \cup B'_\lambda{}^*)$ in R^3 keeping $D_0 \cup D'_0$ fixed, and so the lemma follows.

(Step 1). We may assume that the bands $B_1, \dots, B_\mu, B'_1, \dots, B'_\lambda$ are sufficiently narrow, and so we may suppose, after a (small) ambient isotopy, that the projection of $(\emptyset \cup B_1 \cup \dots \cup B_\mu) \cup (\emptyset' \cup B'_1 \cup \dots \cup B'_\lambda)$ onto R^2 is *regular* in the sense of knot theory; that is, the singularities may be indicated diagrammatically in the usual ways over/under crossings as shown in Figure 1.

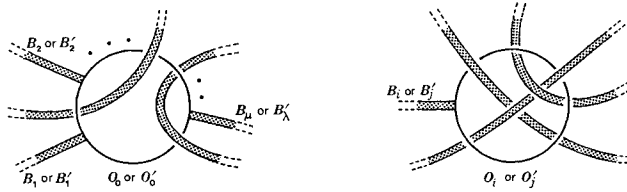


Figure 1

We deform B_1 link-homotopically by changing some crossings of B_1 with $\emptyset$ so that the new B_1 lies under $\emptyset$, see Figure 2(a); and we still denote the new band by B_1 . Inductively, we deform B_i link-homotopically by changing some crossings of B_i with $\emptyset \cup B_1 \cup \dots \cup B_{i-1}$ so that the new B_i lies under $\emptyset \cup B_1 \cup \dots \cup B_{i-1}$, see Figure 2(a) and (b); and we still denote the new band by B_i ($i=2, \dots, \mu$).

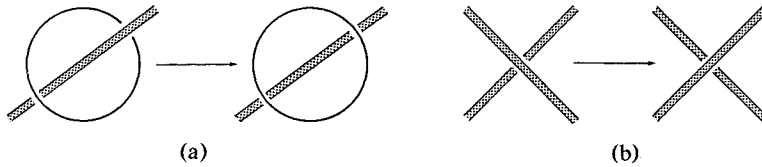


Figure 2

By the same way as that of B_i , we deform B'_1 link-homotopically by changing some crossings of B'_1 with $\emptyset'$ so that the new B'_1 lies under $\emptyset'$; and we still denote it by B'_1 . Inductively, we deform B'_j link-homotopically by changing some crossings of B'_j with $\emptyset' \cup B'_1 \cup \dots \cup B'_{j-1}$ so that the new B'_j lies under $\emptyset' \cup B'_1 \cup \dots \cup B'_{j-1}$ ($j=2, \dots, \lambda$); and we still denote the new band by B'_j .

It will be noticed that $(B_1 \cup \dots \cup B_\mu) \cap ({}^\circ D_0 \cup {}^\circ D_1 \cup \dots \cup {}^\circ D_\mu) = \emptyset$, $(B'_1 \cup \dots \cup B'_\lambda) \cap ({}^\circ D'_0 \cup {}^\circ D'_1 \cup \dots \cup {}^\circ D'_\lambda) = \emptyset$.

(Step 2). In this step, we will remove the intersections $(B_1 \cup \dots \cup B_\mu) \cap (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda)$ and $(B'_1 \cup \dots \cup B'_\lambda) \cap (D_0 \cup D_1 \cup \dots \cup D_\mu)$. We assume that $B_i \cap (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda) \neq \emptyset$. We give a direction on B_i from O_i to O_0 , and let $b_1, b_2, \dots, b_m$ be the segments of $B_i \cap (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda)$ ordered according to the direction, and we may assume that $b_1 \subset B_i \cap D'_j$. We deform $D_i \cup B_i$ in the regular neighborhood $N(D_i \cup \bar{B}_i \cup D'_j)$ isotopically so that $B_i \cup (D'_0 \cup D'_1 \cup \dots \cup D'_\lambda) = b_2 \cup \dots \cup b_m$ as shown in Figure 3, where $\bar{B}_i$ is a subband of B_i cut off by b_1 . Since $(B_1 \cup \dots \cup B_\mu) \cap {}^\circ D_i = \emptyset$, we see that $(B_1 \cup \dots \cup B_\mu) \cap ({}^\circ D_0 \cup {}^\circ D_1 \cup \dots \cup {}^\circ D_\mu) = \emptyset$. On the other hand, some new intersections occur in $(B'_1 \cup \dots \cup B'_\lambda) \cap {}^\circ D'_j$, but these new intersections can be removed by applying the same deformations as that of (Step 1).

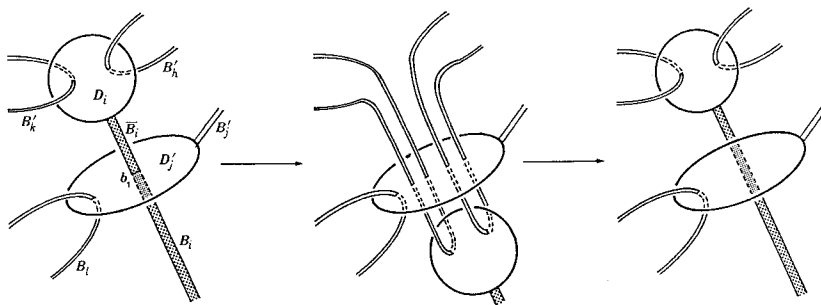


Figure 3

By the repetition of the procedure, we have a link-homotopy of $(\emptyset \cup B_1 \cup \dots \cup B_\mu) \cup (\emptyset' \cup B'_1 \cup \dots \cup B'_\lambda)$ so that $(B_1 \cup \dots \cup B_\mu) \cap ({}^\circ D_0 \cup {}^\circ D_1 \cup \dots \cup {}^\circ D_\mu \cup D'_0 \cup D'_1 \cup \dots \cup D'_\lambda) = \emptyset$, $(B'_1 \cup \dots \cup B'_\lambda) \cap ({}^\circ D'_0 \cup {}^\circ D'_1 \cup \dots \cup {}^\circ D'_\lambda) = \emptyset$. Now it is easy to check that $(\emptyset \cup B_1 \cup \dots \cup B_\mu) \cup (\emptyset' \cup B'_1 \cup \dots \cup B'_\lambda)$ is ambient isotopic to $(\emptyset \cup B_1^* \cup \dots \cup B_\mu^*) \cup (\emptyset' \cup B_1'^* \cup \dots \cup B_\lambda'^*)$ keeping $O_0 \cup O'_0$ fixed, and we complete the proof.

2.2. LEMMA. Let $\tilde{D}_0$ and $\tilde{D}'_0$ be disjoint oriented disks in the hyperplane R^2 of R^3 and let $\tilde{O}_0 = \partial \tilde{D}_0$, $\tilde{O}'_0 = \partial \tilde{D}'_0$, and let $a_1, \dots, a_\alpha$ be mutually disjoint (small) subarcs of $\tilde{O}_0$ and $a'_1, \dots, a'_\beta$ be mutually disjoint (small) subarcs of $\tilde{O}'_0$. Let $b_1, \dots, b_\alpha$, $b'_1, \dots, b'_\beta$ be mutually disjoint simple arcs in R^3 such that $b_i \cap (\tilde{O}_0 \cup \tilde{O}'_0) = \partial b_i \cap \tilde{O}_0 = \partial b_i = \partial a_i$ ($i=1, \dots, \alpha$) and $b'_j \cap (\tilde{O}_0 \cup \tilde{O}'_0) = \partial b'_j \cap \tilde{O}'_0 = \partial b'_j = \partial a'_j$ ($j=1, \dots, \beta$) and let $k_i = a_i \cup b_i$ and $k'_j = a'_j \cup b'_j$.

Let $b_1^*, \dots, b_\alpha^*$, $b_1'^*, \dots, b_\beta'^*$ be mutually disjoint simple arcs in R^2 such that $b_i^* \cap (\tilde{D}_0 \cup \tilde{D}'_0) = \partial b_i^* \cap \tilde{O}_0 = \partial b_i^* = \partial a_i$ ($i=1, \dots, \alpha$) and $b_j'^* \cap (\tilde{D}_0 \cup \tilde{D}'_0) = \partial b_j'^* \cap \tilde{O}'_0 = \partial b_j'^* = \partial a'_j$ ($j=1, \dots, \beta$).

We suppose that

(1) the polyhedron $\mathcal{P} \cup \mathcal{P}' = (k_1 \cup \dots \cup k_\alpha) \cup (k'_1 \cup \dots \cup k'_\beta)$ (=1-link with $\alpha + \beta$ components) is link-homotopic to a trivial 1-link, say $\tilde{\mathcal{O}} \cup \tilde{\mathcal{O}}' = (\tilde{\mathcal{O}}_1 \cup \dots \cup \tilde{\mathcal{O}}_\alpha) \cup (\tilde{\mathcal{O}}'_1 \cup \dots \cup \tilde{\mathcal{O}}'_\beta)$ with $\alpha + \beta$ components in R^3 , as two components polyhedra (that is, there exists a link-homotopy $\{\xi_t\}_{t \in I}$ between $\mathcal{P} \cup \mathcal{P}' = (k_1 \cup \dots \cup k_\alpha) \cup (k'_1 \cup \dots \cup k'_\beta)$ and $\tilde{\mathcal{O}} \cup \tilde{\mathcal{O}}' = (\tilde{\mathcal{O}}_1 \cup \dots \cup \tilde{\mathcal{O}}_\alpha) \cup (\tilde{\mathcal{O}}'_1 \cup \dots \cup \tilde{\mathcal{O}}'_\beta)$ such that $\xi_t(\mathcal{P}) \cap \xi_t(\mathcal{P}') = \emptyset$ for any $t \in I$, $\xi_1(\mathcal{P}) = \xi_1(k_1 \cup \dots \cup k_\alpha) = \tilde{\mathcal{O}} = \tilde{\mathcal{O}}_1 \cup \dots \cup \tilde{\mathcal{O}}_\alpha$ and $\xi_1(\mathcal{P}') = \xi_1(k'_1 \cup \dots \cup k'_\beta) = \tilde{\mathcal{O}}' = \tilde{\mathcal{O}}'_1 \cup \dots \cup \tilde{\mathcal{O}}'_\beta$), and

(2) k_i is contractible in $R^3 - \tilde{\mathcal{O}}_0$ ($i=1, \dots, \alpha$) and k'_j is contractible in $R^3 - \tilde{\mathcal{O}}_0$ ($j=1, \dots, \beta$).

Then, there exists a link-homotopy $\{\tilde{\eta}_t\}_{t \in I}$ between $(\tilde{\mathcal{O}}_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (\tilde{\mathcal{O}}'_0 \cup b'_1 \cup \dots \cup b'_\beta)$ and $(\tilde{\mathcal{O}}_0 \cup b_1^* \cup \dots \cup b_\alpha^*) \cup (\tilde{\mathcal{O}}'_0 \cup b'_1^* \cup \dots \cup b'_\beta^*)$ in R^3 such that $\tilde{\eta}_t(\tilde{\mathcal{O}}_0 \cup \tilde{\mathcal{O}}'_0) = \tilde{\mathcal{O}}_0 \cup \tilde{\mathcal{O}}'_0$ for any $t \in I$, and so $\tilde{\eta}_1(b_i) = b_i^*$ ($i=1, \dots, \alpha$) and $\tilde{\eta}_1(b'_j) = b'^*_j$ ($j=1, \dots, \beta$).

Proof. It is convenient here to drop the tildes and write $D_0, D'_0, O_0, O'_0, O_1, \dots, O_\alpha, O'_1, \dots, O'_\beta$ for $\tilde{D}_0, \tilde{D}'_0, \tilde{\mathcal{O}}_0, \tilde{\mathcal{O}}'_0, \tilde{\mathcal{O}}_1, \dots, \tilde{\mathcal{O}}_\alpha, \tilde{\mathcal{O}}'_1, \dots, \tilde{\mathcal{O}}'_\beta$. We may assume, after a (small) ambient isotopy, that the projection of $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$ onto R^2 is *regular* in the sense of knot theory; i.e. its only singularities are double points in the interiors of 1-simplexes of the 1-complex $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$. These may be indicated diagrammatically in the usual way as over/under crossings.

(Step 1). We deform b_1 link-homotopically by changing some crossings of b_1 with O_0 and itself so that the new b_1 lies under O_0 and the new $k_1 = a_1 \cup b_1$ is unknotted in R^3 ; and we still denote the new arc by b_1 . Inductively, we deform b_i link-homotopically by changing some crossings of b_i with $O_0 \cup b_1 \cup \dots \cup b_{i-1}$ and itself so that the new b_i lies under $O_0 \cup b_1 \cup \dots \cup b_{i-1}$ and the new $k_i = a_i \cup b_i$ is unknotted ($i=2, \dots, \alpha$); and we still denote the new arc by b_i .

By the same ways as that of b_i , we deform b'_j ($j=1, \dots, \beta$) inductively by changing some crossings of b'_j with $O'_0 \cup b'_1 \cup \dots \cup b'_{j-1}$ and itself so that the new b'_j lies under $O'_0 \cup b'_1 \cup \dots \cup b'_{j-1}$ and the new $k'_j = a'_j \cup b'_j$ is unknotted in R^3 ; and we still denote the new arc by b'_j .

Now it should be noted that

(1) the new polyhedron $\mathcal{P} \cup \mathcal{P}' = (k_1 \cup \dots \cup k_\alpha) \cup (k'_1 \cup \dots \cup k'_\beta)$ is also link-homotopic to a trivial 1-link $(O_1 \cup \dots \cup O_\alpha) \cup (O'_1 \cup \dots \cup O'_\beta)$ in R^3 as two components polyhedra,

(2) k_i is contractible in $R^3 - O'_0$ ($i=1, \dots, \alpha$) and k'_j is contractible in $R^3 - O_0$ ($j=1, \dots, \beta$),

(3) the new 1-links $(k_1 \cup \dots \cup k_\alpha \subset R^3)$ and $(k'_1 \cup \dots \cup k'_\beta \subset R^3)$ are trivial, and

(4) $(b_1 \cup \dots \cup b_\alpha) \cap {}^\circ D_0 = \emptyset$, $(b'_1 \cup \dots \cup b'_\beta) \cap {}^\circ D'_0 = \emptyset$.

From this condition (4), we may also assume, after an ambient isotopy keeping $D_0 \cup D'_0$ fixed, that the projection of ${}^\circ b_i$ has no crossings with O_0 ($i = 1, \dots, \alpha$) and the projection of ${}^\circ b'_j$ has no crossings with O'_0 ($j = 1, \dots, \beta$).

(Step 2). In this setep, we will remove the intersections $(b_1 \cup \dots \cup b_\alpha) \cap D'_0$ and $(b'_1 \cup \dots \cup b'_\beta) \cap D_0$ keeping the above conditions (1), (2), (3), (4).

(i) By the conditions (2) and (4), k_1 bounds a *singular* disk, say D_1 , in $R^3 - O'_0$ such that $D_1 \cap D_0 = k_1 \cap O_0 = a_1$, and we may assume that singularities of D_1 are only clasps (see, for example, Smythe [14]). We deform b_1 along the clasps homotopically so that D_1 is non-singular. See Figure 4 below. If

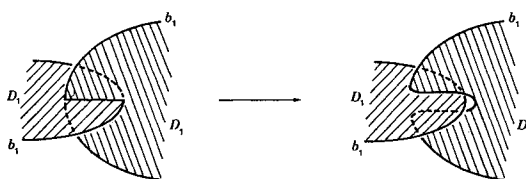


Figure 4

$(b_2 \cup \dots \cup b_\alpha) \cap D_1 \neq \emptyset$, we can deform $b_2 \cup \dots \cup b_\alpha$ link-homotopically, so that $(b_2 \cup \dots \cup b_\alpha) \cap D_1 = \emptyset$ and the conditions (1), (2), (3) and (4) hold. Now $D_1 \cap D'_0 = D_1 \cap {}^\circ D'_0$ consists of a finite number of simple arcs, say $\gamma_1, \dots, \gamma_n$, and simple loops, say $\delta_1, \dots, \delta_m$. These simple arcs are proper on D_1 , and so at least one of these arcs, say γ_1 , cuts off a disk, say Δ_1 , on D_1 such that $\partial \Delta_1 \cap D'_0 = \gamma_1$ and $\partial \Delta_1 \cap a_1 = \emptyset$. By deforming b_1 along Δ_1 isotopically one may obtain a disk D_1 bounded by k_1 which intersects D'_0 in a subcollection of $\gamma_2, \dots, \gamma_n, \delta_1, \dots, \delta_m$. Some new intersections occur in $(b_2 \cup \dots \cup b_\alpha \cup b'_1 \cup \dots \cup b'_\beta) \cap D'_0$, but the intersection $(b'_1 \cup \dots \cup b'_\beta) \cap {}^\circ D'_0$ may be removed by the same way as that of (Step 1).

Repeating the procedure, we have a disk, say D_1 again, with $\partial D_1 = k_1 = a_1 \cup b_1$ such that $D_1 \cap D_0 = k_1 \cap O_0 = a_1$, $D_1 \cap D'_0 = \emptyset$ and $(b_2 \cup \dots \cup b_\alpha) \cap D_1 = \emptyset$, since the simple loops in $D_1 \cap D'_0$ can be easily removed.

(ii) In general, we may assume that there exist mutually disjoint disks $D_1, \dots, D_{i-1}$ in R^3 with $\partial D_v = k_v = a_v \cup b_v$, $D_v \cap D_0 = k_v \cap O_0 = a_v$ ($v = 1, \dots, i-1$), $(D_1 \cup \dots \cup D_{i-1}) \cap D'_0 = \emptyset$ and $(b_i \cup \dots \cup b_\alpha) \cap (D_1 \cup \dots \cup D_{i-1}) = \emptyset$. It follows from the conditions (2), (4) and above properties of $D_1 \cup \dots \cup D_{i-1}$ that k_i bounds a *singular* disk, say D_i , in $R^3 - O'_0$ such that $D_i \cap (D_0 \cup D_1 \cup \dots \cup D_{i-1}) = k_i \cap O_0 = a_i$ and singularities of D_i are only clasps. By the repetition of the procedure in above (i), we have a disk, say D_i again, with $\partial D_i = k_i = a_i \cup b_i$ such that $D_i \cap (D_0 \cup D_1 \cup \dots \cup D_{i-1}) = k_i \cap O_0 = a_i$, $D_i \cap D'_0 = \emptyset$ and $(b_{i+1} \cup \dots \cup b_\alpha) \cap (D_1 \cup \dots \cup D_i) = \emptyset$ for $i = 2, \dots, \alpha$.

After all, we can conclude that there exists a link-homotopy of $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$, which keeps $O_0 \cup O'_0$ fixed, so that the new $(O_0 \cup b_1 \cup \dots \cup$

$b_\alpha) \cup (O'_0 \cup b'_1 \cup \cdots \cup b'_\beta)$ satisfies the conditions (1), (2) and (3) in (Step 1), and

$$(4') \quad (b_1 \cup \cdots \cup b_\alpha) \cap ({}^\circ D_0 \cup D'_0) = \emptyset, (b'_1 \cup \cdots \cup b'_\beta) \cap {}^\circ D'_0 = \emptyset.$$

(iii) Applying the same arguments as above (i) and (ii) to $b'_1, \dots, b'_\beta$, inductively, we have a link-homotopy of $(O_0 \cup b_1 \cup \cdots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \cdots \cup b'_\beta)$, which keeps $O_0 \cup O'_0$ fixed, so that the new $(O_0 \cup b_1 \cup \cdots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \cdots \cup b'_\beta)$ satisfies the conditions (1), (2) and (3) in (Step 1) and

$$(4'') \quad (b_1 \cup \cdots \cup b_\alpha \cup b'_1 \cup \cdots \cup b'_\beta) \cap ({}^\circ D_0 \cup {}^\circ D'_0) = \emptyset.$$

(Step 3). After (Step 2), we have mutually disjoint disks $D'_1, \dots, D'_\beta$ in R^3 such that $\partial D'_j = k'_j = a'_j \cup b'_j$, $D'_j \cap D'_0 = k'_j \cap O'_0 = a'_j$ and $D'_j \cap D_0 = \emptyset$ ($j=1, \dots, \beta$). In general, $(b_1 \cup \cdots \cup b_\alpha) \cap (D'_1 \cup \cdots \cup D'_\beta) \neq \emptyset$, and so we will remove the intersection keeping the above conditions (1), (2), (3) and (4''). The following argument is very similar to that of the preceding (Step 2).

$(b_1)-(i)$: It follows from the conditions (1), (2) and (4'') that $k_1 \cup D_0$ is contractible in $R^3 - (k'_1 \cup D'_0)$, and so there exists a *singular* disk, say D_1^* , in $R^3 - (k'_1 \cup D'_0)$ such that $D_1^* \cap D_0 = k_1 \cup O_0 = a_1$ and singularities of D_1^* are only clasps. We deform b_1 along the clasps so that D_1^* is non-singular by the same way as Figure 4 in (Step 2). If $(b_2 \cup \cdots \cup b_\alpha) \cap D_1^* \neq \emptyset$, we deform $b_2 \cup \cdots \cup b_\alpha$ link-homotopically so that $(b_2 \cup \cdots \cup b_\alpha) \cap D_1^* = \emptyset$ and the conditions (1), (2), (3) and (4'') hold. Now $D_1^* \cap D'_1 = D_1^* \cap {}^\circ D'_1$ consists of a finite number of simple arcs, say $\gamma_1^*, \dots, \gamma_n^*$, and simple loops, say $\delta_1^*, \dots, \delta_m^*$. These arcs are proper on D_1^* , we can choose an innermost one, say γ_1^* , in these arcs; that is, γ_1^* cuts off a disk Δ_1^* in D_1^* such that $\partial \Delta_1^* \cap D'_1 = \gamma_1^*$ and $\partial \Delta_1^* \cap a_1 = \emptyset$. By deforming b_1 along Δ_1^* isotopically we obtain a disk D_1^* bounded by k_1 such that $D_1^* \cap D'_1 \subset \gamma_2^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_m^*$. Some new intersections occur in $(b_2 \cup \cdots \cup b_\alpha \cup b'_2 \cup \cdots \cup b'_\beta) \cap D'_1$, but the intersection $(b'_2 \cup \cdots \cup b'_\beta) \cap D'_1$ may be removed by the same way as that of (Step 1). Repeating the procedure, we have a disk, say again D_1^* , in R^3 with $\partial D_1^* = k_1$ such that $D_1^* \cap D_0 = k_1 \cap O_0 = a_1$, $D_1^* \cap (D'_0 \cup D'_1) = \emptyset$ and so $b_1 \cap (D'_0 \cup D'_1) = \emptyset$.

$(b_1)-(ii)$: In general, we suppose that there exist mutually disjoint disks $D'_1, \dots, D'_{j-1}$ in R^3 such that $\partial D'_v = k'_v = a'_v \cup b'_v$, $D'_v \cap D'_0 = k'_v \cap O'_0 = a'_v$ ($v=1, \dots, j-1$), $(D'_1 \cup \cdots \cup D'_{j-1}) \cap D_0 = \emptyset$ and $b_1 \cap (D'_1 \cup \cdots \cup D'_{j-1}) = \emptyset$. Since $b'_j \cap (D'_0 \cup D'_1 \cup \cdots \cup D'_{j-1}) = \partial b'_j \cap O'_0 = \partial b'_j$, from the condition (3) we have a disk, say D'_j , in R^3 with $\partial D'_j = k'_j$ such that $D'_j \cap (D_0 \cup D'_0 \cup D'_1 \cup \cdots \cup D'_{j-1}) = k'_j \cap O'_0 = a'_j$. From the conditions (1), (2) and (4''), $k_1 \cup D_0$ is contractible in $R^3 - (k'_j \cup D'_0 \cup D'_1 \cup \cdots \cup D'_{j-1})$, and thus from (4'') there exists a *singular* disk, again say D_1^* , in $R^3 - (k'_j \cup D'_0 \cup D'_1 \cup \cdots \cup D'_{j-1})$ such that $D_1^* \cap D_0 = k_1 \cap O_0 = a_1$ and singularities of D_1^* are only clasps. Applying the same argument as above $(b_1)-(i)$, we have a disk, say again D_1^* , with $\partial D_1^* = k_1$ such that $D_1^* \cap D_0 = k_1 \cap O_0 = a_1$, $D_1^* \cap (D'_0 \cup D'_1 \cup \cdots \cup D'_j) = \emptyset$ for $j=2, \dots, \alpha$.

Therefore, we conclude that there exists a link-homotopy of $(O_0 \cup b_1 \cup \cdots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \cdots \cup b'_\beta)$, which keeps $O_0 \cup O'_0$ fixed, so that the new $(O_0 \cup b_1 \cup \cdots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \cdots \cup b'_\beta)$ satisfies the conditions (1), (2), (3) in (Step 1) and (4'') in

(Step 2)–(iii), and

(5) there exist mutually disjoint disks, say again $D_1^*, D'_1, \dots, D'_\beta$, in R^3 with $\partial D_1^* = k_1$, $\partial D'_j = k'_j$ ($j=1, \dots, \beta$) such that $D_1^* \cap (D_0 \cup D'_0) = k_1 \cap O_0 = a_1$, $D'_j \cap (D_0 \cup D'_0) = k'_j \cap O'_0 = a'_j$ and $D_1^* \cap (b_2 \cup \dots \cup b_\alpha) = \emptyset$.

REMARK. The set of disks $D'_1, \dots, D'_\beta$ in (5) is different from the set of disks $D'_1, \dots, D'_\beta$ mentioned in the first paragraph of (Step 3). In fact, at the beginning of (b_1) –(i), $k_1 \cup D_0$ is not always contractible in $R^3 - (k'_1 \cup \dots \cup k'_\beta \cup D'_0)$ (Recall the Brunnian rings [11]).

(Step 4). After (Step 3), we may assume, inductively, that

(5') there exist mutually disjoint disks, say $D_1^*, \dots, D_{i-1}^*, D'_1, \dots, D'_\beta$, in R^3 with $\partial D_v^* = k_v$ ($v=1, \dots, i-1$), $\partial D'_j = k'_j$ ($j=1, \dots, \beta$) such that $D_v^* \cap (D_0 \cup D'_0) = k_v \cap O_0 = a_v$, $D'_j \cap (D_0 \cup D'_0) = k'_j \cap O'_0 = a'_j$ and $(D_1^* \cup \dots \cup D_{i-1}^*) \cap (b_i \cup \dots \cup b_\alpha) = \emptyset$.

(b_i) –(i): From the conditions (2), (4'') and (5'), we see that $k_i \cup D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*$ is contractible in $R^3 - (k'_1 \cup D'_0)$. Therefore, we have a *singular* disk, say D_i^* , in $R^3 - (k'_1 \cup D'_0)$ such that $D_i^* \cap (D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*) = k_i \cap O_0 = a_i$ and singularities of D_i^* are only clasps. Applying the same argument as (b_1) –(i), we have a disk, say again D_i^* , with $\partial D_i^* = k_i$ such that $D_i^* \cap (D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*) = k_i \cap O_0 = a_i$, $D_i^* \cap (D'_0 \cup D'_1) = \emptyset$ and so $b_i \cap (D'_0 \cup D'_1) = \emptyset$.

(b_i) –(ii): Inductively, we may assume that there exist mutually disjoint disks, say $D'_1, \dots, D'_{j-1}$ and D'_j , in R^3 with $\partial D'_v = k'_v$, $D'_v \cap D'_0 = k'_v \cap O'_0 = a'_v$ ($v=1, \dots, j$) such that $(D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*) \cap (D'_0 \cup D'_1 \cup \dots \cup D'_j) = \emptyset$, and also assume that $k_i \cup D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*$ is contractible in $R^3 - (k'_j \cup D'_0 \cup D'_1 \cup \dots \cup D'_{j-1})$. Thus we have a *singular* disk, say D_i^* , in $R^3 - (k'_j \cup D'_0 \cup D'_1 \cup \dots \cup D'_{j-1})$ such that singularities of D_i^* are only clasps. Applying the same argument as (b_1) –(ii), we have a disk, say again D_i^* , with $\partial D_i^* = k_i$ such that $D_i^* \cap (D_0 \cup D_1^* \cup \dots \cup D_{i-1}^*) = k_i \cap O_0 = a_i$, $D_i^* \cap (D'_0 \cup D'_1 \cup \dots \cup D'_j) = \emptyset$ and $(D_1^* \cup \dots \cup D_i^*) \cap (b_{i+1} \cup \dots \cup b_\alpha) = \emptyset$ for $i=2, \dots, \alpha$ and $j=1, \dots, \beta$.

Thus, in the long run, we conclude that there exists a link-homotopy of $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$, which keeps $O_0 \cup O'_0$ fixed, so that the new $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$ satisfies the conditions (1), (2), (3) in (Step 1) and (4'') in (Step 2)–(iii), and

(5'') there exist mutually disjoint disks, say $D_1^*, \dots, D_\alpha^*, D'_1, \dots, D'_\beta$, in R^3 with $\partial D_i^* = k_i$ ($i=1, \dots, \alpha$), $\partial D'_j = k'_j$ ($j=1, \dots, \beta$) such that $D_i^* \cap (D_0 \cup D'_0) = k_i \cap O_0 = a_i$, $D'_j \cap (D_0 \cup D'_0) = k'_j \cap O'_0 = a'_j$.

Now, it is easy to obtain a required link-homotopy between $(O_0 \cup b_1 \cup \dots \cup b_\alpha) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$ and $(O_0 \cup b_1^* \cup \dots \cup b_\alpha^*) \cup (O'_0 \cup b'_1 \cup \dots \cup b'_\beta)$, and we complete the proof of Lemma 2.2.

2.3. LEMMA. Let $\tilde{D}_0$ and $\tilde{D}'_0$ be disjoint oriented disks in the hyperplane R^2

of R^3 and let $\tilde{O}_0 = \partial \tilde{D}_0$, $\tilde{O}'_0 = \partial \tilde{D}'_0$, and let $a_1, \dots, a_\alpha$ be mutually disjoint (small) subarcs of $\tilde{O}_0$ and $a'_1, \dots, a'_\beta$ be mutually disjoint (small) subarcs of $\tilde{O}'_0$. Let $\tilde{B}_1, \dots, \tilde{B}_\alpha, \tilde{B}'_1, \dots, \tilde{B}'_\beta$ be mutually disjoint oriented bands in R^3 such that each $\tilde{B}_i$ spans $\tilde{O}_0$ at two subarcs of a_i ($i=1, \dots, \alpha$) and each $\tilde{B}'_j$ spans $\tilde{O}'_0$ at two subarcs of a'_j ($j=1, \dots, \beta$), and we set

$$\begin{aligned} h(\tilde{O}_0 \cup \tilde{O}'_0; \tilde{B}_1, \dots, \tilde{B}_\alpha, \tilde{B}'_1, \dots, \tilde{B}'_\beta) &= h(\tilde{O}_0; \tilde{B}_1, \dots, \tilde{B}_\alpha) \cup h(\tilde{O}'_0; \tilde{B}'_1, \dots, \tilde{B}'_\beta) \\ &= (\tilde{O}_0^* \cup \tilde{O}_1 \cup \dots \cup \tilde{O}_\alpha) \cup (\tilde{O}'_0^* \cup \tilde{O}'_1 \cup \dots \cup \tilde{O}'_\beta), \end{aligned}$$

where $\tilde{O}_i$ is the component of $h(\tilde{O}_0; \tilde{B}_1, \dots, \tilde{B}_\alpha)$ with $\tilde{O}_i \cap a_i \neq \emptyset$ ($i=1, \dots, \alpha$) and $\tilde{O}'_j$ is the component of $h(\tilde{O}'_0; \tilde{B}'_1, \dots, \tilde{B}'_\beta)$ with $\tilde{O}'_j \cap a'_j \neq \emptyset$ ($j=1, \dots, \beta$).

Let $B_1^*, \dots, B_\alpha^*, B_1'^*, \dots, B_\beta'^*$ be mutually disjoint oriented bands in R^2 such that B_i^* spans $\tilde{O}_0$ with $B_i^* \cap \tilde{D}_0 = B_i^* \cap \tilde{O}_0 = \tilde{B}_i \cap \tilde{O}_0$ ($i=1, \dots, \alpha$) and $B_j'^*$ spans $\tilde{O}'_0$ with $B_j'^* \cap \tilde{D}'_0 = B_j'^* \cap \tilde{O}'_0 = \tilde{B}'_j \cap \tilde{O}'_0$ ($j=1, \dots, \beta$).

We suppose that

(a) $\tilde{O}_0^* \cup \tilde{O}_1 \cup \dots \cup \tilde{O}_\alpha \cup \tilde{O}'_0^* \cup \tilde{O}'_1 \cup \dots \cup \tilde{O}'_\beta$ is link-homotopic to a trivial 1-link with $\alpha + \beta + 2$ components in R^3 , and

(b) $\tilde{O}_1 \cup \dots \cup \tilde{O}_\alpha \cup \tilde{O}'_1 \cup \dots \cup \tilde{O}'_\beta$ is a trivial 1-link with $\alpha + \beta$ components.

Then, there exists a link-homotopy $\{\tilde{\eta}_t\}_{t \in I}$ between $(\tilde{O}_0 \cup \tilde{B}_1 \cup \dots \cup \tilde{B}_\alpha) \cup (\tilde{O}'_0 \cup \tilde{B}'_1 \cup \dots \cup \tilde{B}'_\beta)$ and $(\tilde{O}_0 \cup B_1^* \cup \dots \cup B_\alpha^*) \cup (\tilde{O}'_0 \cup B_1'^* \cup \dots \cup B_\beta'^*)$ such that $\tilde{\eta}_t(\tilde{O}_0 \cup \tilde{O}'_0) = \tilde{O}_0 \cup \tilde{O}'_0$ for any $t \in I$ and so $\tilde{\eta}_1(\tilde{B}_i) = B_i^*$ ($i=1, \dots, \alpha$) and $\tilde{\eta}_1(\tilde{B}'_j) = B_j'^*$ ($j=1, \dots, \beta$).

Proof. Let b_i (resp. b'_j) be the associated connecting arc of the band $\tilde{B}_i$ (resp. $\tilde{B}'_j$) (i.e. b_i (resp. b'_j) is the core of $\tilde{B}_i$ (resp. $\tilde{B}'_j$), Cf. Goldsmith [5, Def. 1.5]), $i=1, \dots, \alpha$ (resp. $j=1, \dots, \beta$). Let k_i (resp. k'_j) be the simple loop in $a_i \cup b_i$ (resp. $a'_j \cup b'_j$).

From our definition, $(k_1 \cup \dots \cup k_\alpha) \cup (k'_1 \cup \dots \cup k'_\beta)$ is isotopic to $(\tilde{O}_1 \cup \dots \cup \tilde{O}_\alpha) \cup (\tilde{O}'_1 \cup \dots \cup \tilde{O}'_\beta)$ in R^3 , and so $(k_1 \cup \dots \cup k_\alpha) \cup (k'_1 \cup \dots \cup k'_\beta)$ satisfies the assumption (1) in Lemma 2.2.

If we assign a suitable orientation to every loop of $\tilde{O}_0^*, \tilde{O}_1, \dots, \tilde{O}_\alpha, \tilde{O}'_0^*, \tilde{O}'_1, \dots, \tilde{O}'_\beta, k_1, \dots, k_\alpha, k'_1, \dots, k'_\beta, \tilde{O}_0$ and $\tilde{O}'_0$, then the linking number $lk(k_i, \tilde{O}_0^*) = lk(\tilde{O}_i, \tilde{O}_0^*) = 0$, $lk(k_i, \tilde{O}'_j) = lk(\tilde{O}_i, \tilde{O}'_j) = 0$, $lk(k'_j, \tilde{O}_0^*) = lk(\tilde{O}'_j, \tilde{O}_0^*) = 0$ for any $i=1, \dots, \alpha$ and $j=1, \dots, \beta$. Moreover, as $\mathbb{Z}_2$ -cycles, $\tilde{O}_0^* + \tilde{O}_i$ is homologous to $\tilde{O}_0$ in $R^3 - (k_1 \cup \dots \cup k_\alpha \cup k'_1 \cup \dots \cup k'_\beta)$, and $\tilde{O}'_0^* + \tilde{O}'_j$ is homologous to $\tilde{O}'_0$ in $R^3 - (k_1 \cup \dots \cup k_\alpha \cup k'_1 \cup \dots \cup k'_\beta)$. Therefore, we conclude that $lk(k_i, \tilde{O}_0) = lk(k'_j, \tilde{O}_0) = 0$ for $i=1, \dots, \alpha$ and $j=1, \dots, \beta$. Since $\tilde{O}_0$ and $\tilde{O}'_0$ are unknotted, k_i is contractible in $R^3 - \tilde{O}'_0$ ($i=1, \dots, \alpha$) and k'_j is contractible in $R^3 - \tilde{O}_0$ ($j=1, \dots, \beta$), which is the assumption (2) in Lemma 2.2.

Now, Lemma 2.3 is an immediate consequence of Lemma 2.2, because twistings of bands $\tilde{B}_i$ and $\tilde{B}'_j$ can be removed by link-homotopy in the neighborhoods of $\tilde{B}_i$ and $\tilde{B}'_j$ keeping the ends $\tilde{B}_i \cap a_i$ and $\tilde{B}'_j \cap a'_j$ fixed.

3. Proof of Theorem 1.4.

For clarity, we divide the proof of Theorem 1.4 into three steps.

(Step 1). Since the 2-sphere is compact, locally knotted points (see [15, 1.14], = singularities [4]) of L is finite. We choose a tree, say T_i , on S_i^2 ($i=1, 2$) such that T_i contains the all locally knotted points of L . Then, we have a disk-pair $(L(N(T_i; S_i^2)) \subset N(L(T_i); S^4))$, and by Fox-Milnor's theorem [4] we see the 1-knot $(L(\partial N(T_i; S_i^2)) \subset \partial N(L(T_i); S^4))$ is a slice knot. Thus, there exists a locally flat proper disk, say E_i , in $N(L(T_i); S^4)$ with $\partial E_i = L(\partial N(T_i; S_i^2))$, and we have a locally flat 2-sphere $K_i = L(S_i^2 - {}^\circ N(T_i; S_i^2)) \cup E_i$ in S^4 . By the Alexander's trick, we conclude that $L(S_1^2 \cup S_2^2)$ is isotopic to $K_1 \cup K_2$. Therefore, we may assume that L is locally flat; and from now on we shall discuss the image $(L(S_1^2) \cup L(S_2^2) \subset S^4) = (K \cup K' \subset S^4)$.

(Step 2). By Proposition 1.1, we assume that $K \cup K'$ is in the normal form in R^4 ; $S^4 = R^4 \cup \{\infty\}$. Let $\mathcal{O} \cup \mathcal{O}' \rightarrow \mathcal{K} \cup \mathcal{K}' \rightarrow \tilde{\mathcal{O}} \cup \tilde{\mathcal{O}'}$ be the sequence of oriented 1-links whose closed realizing surface is $K \cup K'$, such that K (resp. K') is a closed realizing surface of $\mathcal{O} \rightarrow \mathcal{K} \rightarrow \tilde{\mathcal{O}}$ (resp. $\mathcal{O}' \rightarrow \mathcal{K}' \rightarrow \tilde{\mathcal{O}'}$), where $\mathcal{O}$, $\mathcal{O}'$, $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}'}$ are trivial 1-links corresponding to $K \cap R^3[-3]$, $K' \cap R^3[-3]$, $K \cap R^3[3]$ and $K' \cap R^3[3]$, respectively, $\mathcal{K}$ and $\mathcal{K}'$ are 1-knots corresponding to the equatorial 1-knots $K \cap R^3[0]$ and $K' \cap R^3[0]$, respectively (We refer the reader to §1 of [7]). We set $\mathcal{O} = O_0 \cup O_1 \cup \dots \cup O_\mu$, $\mathcal{O}' = O'_0 \cup O'_1 \cup \dots \cup O'_\lambda$, $\tilde{\mathcal{O}} = \tilde{O}_0 \cup \tilde{O}_1 \cup \dots \cup \tilde{O}_\alpha$ and $\tilde{\mathcal{O}'} = \tilde{O}'_0 \cup \tilde{O}'_1 \cup \dots \cup \tilde{O}'_\beta$.

Let $D_0, D_1, \dots, D_\mu, D'_0, D'_1, \dots, D'_\lambda$ be mutually disjoint disks in $R^2 \subset R^3$ which are oriented coherently with that of R^2 , and let $O_i = \partial D_i$ ($i=0, 1, \dots, \mu$) and $O'_j = \partial D'_j$ ($j=0, 1, \dots, \lambda$). We may assume that $\tilde{\mathcal{O}} = O_0 \cup O_1 \cup \dots \cup O_\mu$ and $\mathcal{O}' = O'_0 \cup O'_1 \cup \dots \cup O'_\lambda$, and that $\mathcal{K} = h(\mathcal{O}; B_1, \dots, B_\mu)$, $\mathcal{K}' = h(\mathcal{O}'; B'_1, \dots, B'_\lambda)$ and $\tilde{\mathcal{O}} = h(\mathcal{K}; \tilde{B}_1, \dots, \tilde{B}_\alpha)$, $\tilde{\mathcal{O}'} = h(\mathcal{K}'; \tilde{B}'_1, \dots, \tilde{B}'_\beta)$; that is, $\mathcal{K}$ (resp. $\mathcal{K}'$) is obtained from $\mathcal{O}$ (resp. $\mathcal{O}'$) by μ -fusion along $\{B_1, \dots, B_\mu\}$ (resp. λ -fusion along $\{B'_1, \dots, B'_\lambda\}$), and $\tilde{\mathcal{O}}$ (resp. $\tilde{\mathcal{O}'}$) is obtained from $\mathcal{K}$ (resp. $\mathcal{K}'$) by α -fission along $\tilde{\mathcal{B}} = \{\tilde{B}_1, \dots, \tilde{B}_\alpha\}$ (resp. β -fission along $\tilde{\mathcal{B}'} = \{\tilde{B}'_1, \dots, \tilde{B}'_\beta\}$). By [7, Lemma 1.14] and its proof (or [15, Lemma 2.5] and its proof), we may also assume that

- (i) B_i spans O_i and O_0 ($i=1, \dots, \mu$) and B'_j spans O'_j and O'_0 ($j=1, \dots, \lambda$),
- (ii) there exist mutually disjoint (small) abuarcs $a_1, \dots, a_\alpha$ of $O_0 - (B_1 \cup \dots \cup B_\mu)$ and mutually disjoint (small) subarcs $a'_1, \dots, a'_\beta$ of $O'_0 - (B'_1 \cup \dots \cup B'_\lambda)$ such that $\tilde{B}_i$ spans $\mathcal{K} = h(\mathcal{O}; B_1, \dots, B_\mu)$ at two subarcs of a_i and $\tilde{B}'_j$ spans $\mathcal{K}' = h(\mathcal{O}'; B'_1, \dots, B'_\lambda)$ at two subarcs of a'_j ,
- (iii) $(B_1 \cup \dots \cup B_\mu \cup B'_1 \cup \dots \cup B'_\lambda) \cap (\tilde{B}_1 \cup \dots \cup \tilde{B}_\alpha \cup \tilde{B}'_1 \cup \dots \cup \tilde{B}'_\beta) = \emptyset$.

(Step 3). By Lemma 2.1, there exists a link-homotopy, say $\{\eta_s\}_{s \in I}$, of the polyhedron $(\mathcal{O} \cup B_1 \cup \dots \cup B_\mu) \cup (\mathcal{O}' \cup B'_1 \cup \dots \cup B'_\lambda)$ in R^3 so that

- (0) $\eta_s(O_0 \cup O'_0) = O_0 \cup O'_0$ for any $s \in I$, and

(i) $\eta_1((\emptyset \cup B_1 \cup \cdots \cup B_\mu) \cup (\mathcal{O}' \cup B'_1 \cup \cdots \cup B'_\lambda))$ is contained in R^2 . In particular, we may assume that

(ii) $\eta_s(B_1 \cup \cdots \cup B_\mu) \cup (a_1 \cup \cdots \cup a_\alpha) = \emptyset$, $\eta_s(B'_1 \cup \cdots \cup B'_\lambda) \cup (a'_1 \cup \cdots \cup a'_\beta) = \emptyset$ for any $s \in I$,

because the first link-homotopy used in (Step 1) of the proof of Lemma 2.1 can be modified by a suitable isotopy deformation as illustrated in Figure 5.

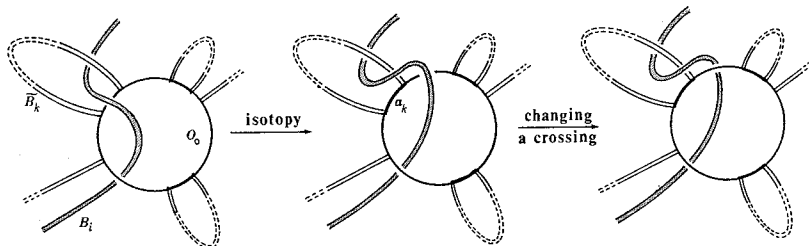


Figure 5

It follows from (ii) and (iii) of (Step 2) that the restriction $\{\eta_s\}_{s \in I}$ on $\tilde{\mathcal{B}} \cup \tilde{\mathcal{B}}' \cup (a_1 \cup \cdots \cup a_\alpha \cup a'_1 \cup \cdots \cup a'_\beta)$ is an isotopy keeping $a_1 \cup \cdots \cup a_\alpha \cup a'_1 \cup \cdots \cup a'_\beta$ fixed. It should be noted that $\eta_1(\mathcal{K} \cup \mathcal{K}') = \eta_1(h(\emptyset; B_1, \dots, B_\mu) \cup h(\mathcal{O}'; B'_1, \dots, B'_\lambda))$ is a trivial 1-link with 2 components, and $\eta_1(\tilde{B}_1), \dots, \eta_1(\tilde{B}_\alpha)$, $\eta_1(\tilde{B}'_1), \dots, \eta_1(\tilde{B}'_\beta)$ are mutually disjoint bands satisfying the assumptions in Lemma 2.3, provided that we replace $\eta_1(\mathcal{K}) \cup \eta_1(\mathcal{K}')$ with $\tilde{\mathcal{O}}_0 \cup \tilde{\mathcal{O}}'_0$ and $\eta_1(\tilde{B}_1), \dots, \eta_1(\tilde{B}_\alpha)$, $\eta_1(\tilde{B}'_1), \dots, \eta_1(\tilde{B}'_\beta)$ with $\tilde{B}_1, \dots, \tilde{B}_\alpha$, $\tilde{B}'_1, \dots, \tilde{B}'_\beta$. By Lemma 2.3, there exists a link-homotopy, say $\{\tilde{\eta}_s\}_{s \in I}$, of the polyhedron $(\eta_1(\mathcal{K}) \cup \eta_1(\tilde{B}_1) \cup \cdots \cup \eta_1(\tilde{B}_\alpha)) \cup (\eta_1(\mathcal{K}') \cup \eta_1(\tilde{B}'_1) \cup \cdots \cup \eta_1(\tilde{B}'_\beta))$ in R^3 such that

(0) $\tilde{\eta}_s(\eta_1(\mathcal{K} \cup \mathcal{K}')) = \eta_1(\mathcal{K} \cup \mathcal{K}')$ for any $s \in I$, and

(i) $\tilde{\eta}_1((\eta_1(\mathcal{K}) \cup \eta_1(\tilde{B}_1) \cup \cdots \cup \eta_1(\tilde{B}_\alpha)) \cup (\eta_1(\mathcal{K}') \cup \eta_1(\tilde{B}'_1) \cup \cdots \cup \eta_1(\tilde{B}'_\beta)))$ is contained in R^2 .

New we can easily construct a required link-homotopy, say $\{\theta_s\}_{s \in I}$, between $K \cup K'$ and a trivial 2-link by extending $\{\eta_s\}_{s \in I}$ and $\{\tilde{\eta}_s\}_{s \in I}$ naturally for each level $(K \cup K') \cup R^3[t]$, $-3 \leq t \leq 3$. In fact, we define $\{\theta_s\}_{s \in I}$ as follows: under identifying $R^3[t]$ with R^3 ,

$$\theta_s((K \cup K') \cap R^3[t]) = \begin{cases} \eta_{2s}((K \cup K') \cap R^3[t]) & \text{for } 0 \leq s \leq 1/2, -3 \leq t \leq 3, \\ \tilde{\eta}_{2s-1}(\eta_1(K \cup K') \cap R^3[t]) & \text{for } 1/2 \leq s \leq 1, 1 \leq t \leq 3, \\ \eta_1((K \cup K') \cap R^3[t]) & \text{for } 1/2 \leq s \leq 1, -3 \leq t \leq 1. \end{cases}$$

(See Figure 6, which shows the $\{\theta_s\}_{s \in I}$ schematically). By Theorem 1.2, or directly, we can easily check that $\theta_1(K \cup K')$ is a trivial 2-link.

This completes the proof of Theorem 1.4.

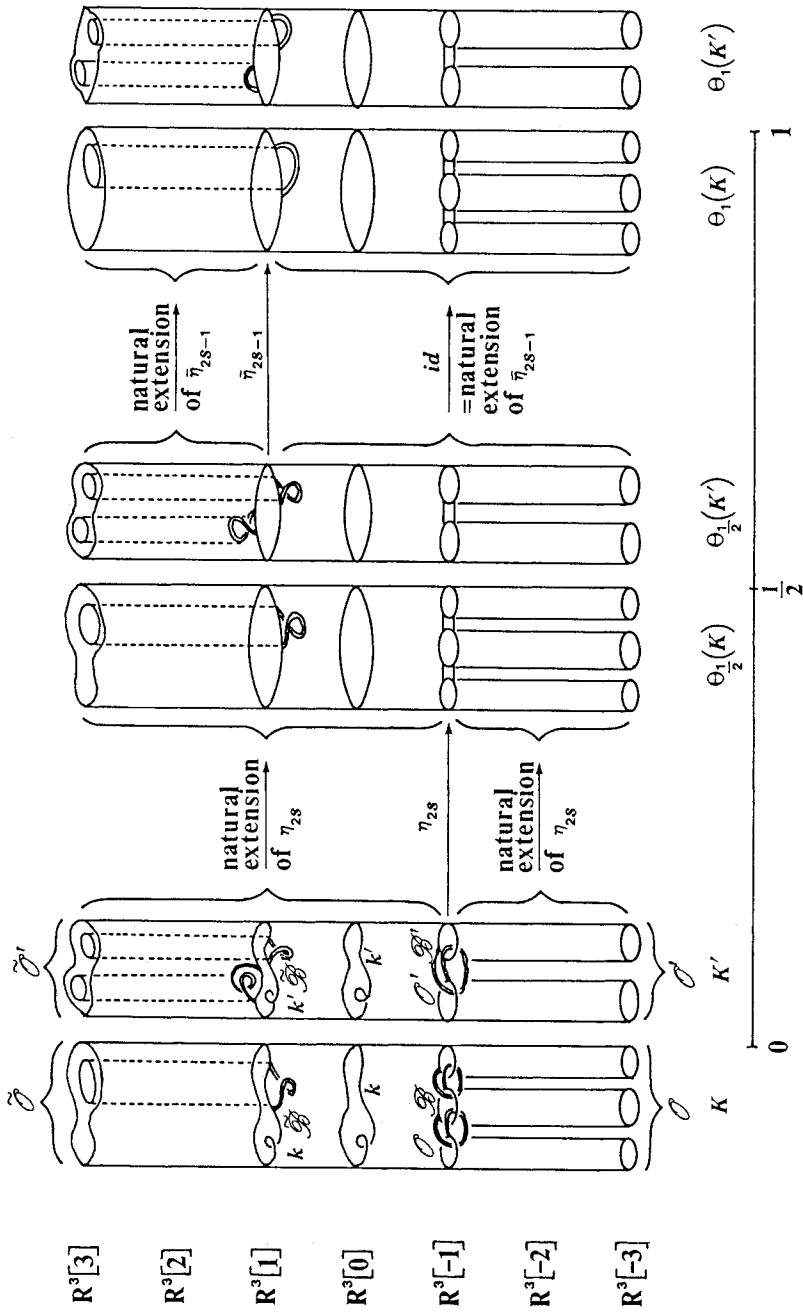


Figure 6

4. Remarks and Questions

The following Question is also proposed by D. Rolfsen [12]:

4.1. QUESTION *Given a 2-link $L: S_1^2 \cup S_2^2 \rightarrow S^4$, is there a homotopy of $L|_{S_1^2}$ in $S^4 - L(S_2^2)$, taking it to a trivial 2-knot in S^4 ?*

It is easy to see that an affirmative answer to this Question implies Theorem 1.4, but we do not know the answer. The following is a partial solution to this Question.

4.2. THEOREM *Let $(K \cup K' \subset S^4)$ be a 2-link. If $(K \subset S^4)$ is a ribbon 2-knot, then there exists a homotopy $\{\theta_s\}_{s \in I}$ of K in $S^4 - K'$ such that $\theta_1(K)$ is a trivial 2-knot in S^4 .*

Sketch Proof. By [7, Cor. 2, 18] and the definition of a ribbon 2-knot (Cf. Yanagawa [17, Theorem (5.3)], [7, Def. 4.1], [15, Def. 2.12]), we may assume that $K \cup K'$ is in the normal form in R^4 , $S^4 = R^4 \cup \{\infty\}$, such that K is simple symmetric with respect to $R^3[0]$. Let $\mathcal{O} \cup \mathcal{O}' \rightarrow \mathcal{K} \cup \mathcal{K}' \rightarrow \tilde{\mathcal{O}} \cup (\tilde{\mathcal{O}}' \cup \mathcal{K}_1' \cup \cdots \cup \mathcal{K}_\nu')$ be the sequence of oriented 1-links whose realizing surface is $K \cup K'$, such that K is a closed realizing surface of the sequence $\mathcal{O} \rightarrow \mathcal{K} \rightarrow \tilde{\mathcal{O}}$. We suppose that $\mathcal{O} = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_\mu$, $\tilde{\mathcal{O}} = \tilde{\mathcal{O}}_0 \cup \tilde{\mathcal{O}}_1 \cup \cdots \cup \tilde{\mathcal{O}}_\alpha$, $\mathcal{K} = h(\mathcal{O}; B_1, \dots, B_\mu)$, $\tilde{\mathcal{O}} = h(\mathcal{K}; \tilde{B}_1, \dots, \tilde{B}_\alpha)$, $\mathcal{K}' = h(\mathcal{O}'; B_1', \dots, B_\lambda')$ and $\tilde{\mathcal{O}}' \cup \mathcal{K}_1' \cup \cdots \cup \mathcal{K}_\nu' = h(\mathcal{K}'; \tilde{B}_1', \dots, \tilde{B}_\beta')$. Then the simple symmetry of K implies that the sequence $\mathcal{O} \rightarrow \mathcal{K}$ is precisely the converse of the sequence $\mathcal{K} \rightarrow \tilde{\mathcal{O}}$, and so $\mu = \alpha$ and we may assume that $B_i = \tilde{B}_i$ as sets ($i = 1, \dots, \mu$).

Let $\{\eta_s\}_{s \in I}$ be a homotopy of the polyhedron $\mathcal{O} \cup B_1 \cup \cdots \cup B_\mu (= \mathcal{O} \cup \tilde{B}_1 \cup \cdots \cup \tilde{B}_\mu)$ in $R^3 - (\mathcal{O}' \cup B_1' \cup \cdots \cup B_\lambda' \cup \tilde{B}_1' \cup \cdots \cup \tilde{B}_\beta')$ obtained from the following sequence of homotopic deformations: We deform $B_1 = \tilde{B}_1$ homotopically by changing some crossings of B_1 with $\mathcal{O}$ so that the new B_1 lies under $\mathcal{O}$, as the same way as that of (Step 1) in the proof of Lemma 2.1. Inductively, we deform $B_i = \tilde{B}_i$ homotopically by changing some crossings of B_i with $\mathcal{O} \cup B_1 \cup \cdots \cup B_{i-1}$ so that B_i lies under $\mathcal{O} \cup B_1 \cup \cdots \cup B_{i-1}$ ($i = 2, \dots, \mu$).

We can construct now a required homotopy $\{\theta_s\}_{s \in I}$ of K in $S^4 - K'$ by the same way as that of (Step 3) in the proof of Theorem 1.4. In fact, we define $\{\theta_s\}_{s \in I}$ as follows: under identifying $R^3[t]$ with R^3 ,

$$\theta_s(K \cap (R^3[t] - K')) = \eta_s(K \cap (R^3[t] - K'))$$

for $0 \leq s \leq 1$ and $-3 \leq t \leq 3$. It is easy to see that $\theta_1(K)$ is a trivial 2-knot in S^4 by Theorem 1.2.

After proving Theorem 1.4, the following question come to mind:

4.3. QUESTION *Is every 2-link $L: \sum_{i=1}^c S_i^2 \rightarrow S^4$ with $c \geq 3$ link-homotopic to a trivial 2-link?*

Lemma 2.1 is valid for $c \geq 3$ components polyhedron, but Lemma 2.2 (and so Lemma 2.3) can not be proved for $c \geq 3$ components polyhedron by our method, and Question 4.3 is still open.

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