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(Citation)
Kobe Journal of Mathematics,4(2):209-217

(Issue Date)
1988-02

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0034668

(URL)
https://hdl.handle.net/20.500.14094/E0034668

YANAGIHARA, H.
KOBE J. MATH.,
4 (1987) 209-217

ON SEMINORMAL LOCAL RINGS AND MULTICROSS SINGULARITIES

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Dedicated to Professor Masayoshi Nagata on his 60-th birthday

(Received June 26, 1987)

In [2] E. D. Davis gave a characerization for a reduced local ring A of dimension 1 to be seminormal in terms of the multiplicity $e(A)$, the embedded dimension emdim (A) and the associated graded ring $G(A)$ of A. Precisely, he showed that A is seminormal if and only if $e(A)=\operatorname{emdim}(A)$ and $G(A)$ is reduced. Our first aim in this paper is to generalize this result in higher dimensional cases in Theorem 1 and to give a geometric interpretation of our result in Theorem 2. The another purpose of this paper is to define a generalized multicross point of an algebraic scheme over an algebraically closed field, which may be considered as a generalized notion of an ordinary multiple point of an algebraic curve. We shall give a characterization of such singularities in terms of multiplicities and projectivized tangent cones in Theorem 3 which was already given in the case of curves by F. Orecchia in his paper [6]. Lastly we shall see a relation between generalized multicross singularities and seminormal ones in Theorem 4.

All the rings in this paper are commutative with identity, and the notations and terminology on commutative ring theory and algebraic geometry follow those of [2], [3], [4], [7] and [8] freely.

1. Seminormal local rings

Let A be a semilocal ring with Jacobson radical J. Then through this paper we denote by $G(A)$ and $G(A)_+$ the associated graded ring $\sum_{n=0}^{\infty} J^n/J^{n+1}$ with respect to J and its positive part $\sum_{n=1}^{\infty} J^n/J^{n+1}$, respectively.

First we give a result for higher dimensional cases corresponding to Proposition 2.2 in [6] where 1-dimensional cases are treated.

PROPOSITION 1. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and B a semilocal ring containing A. Assume that B is a finite A-module and consider the following three conditions:*

(i) *$G(A)$ is reduced.*

(ii) *$J^r \cap \mathfrak{m}^s = \mathfrak{m}^r$ for any integers r, s with $r > s \geqq 0$, where J is the Jacobson radical of B.*

(iii) *The homomorphism $G(A) \to G(B)$ of graded rings induced by the* *natural homomorphism* $\mathfrak{m}^n/\mathfrak{m}^{n+1}\to J^n/J^{n+1}$ *is injective.*

Then (ii) *and* (iii) *are equivalent to each other, and if the conductor* $\mathfrak{c}=A:_A B$ *of* A *in* B *is* $\mathfrak{m}$*-primary,* (i) *implies* (ii) *and* (iii).

PROOF. The first assertion is clear. If $\mathfrak{c}$ is $\mathfrak{m}$-primary, then there is a positive integer s such that $\mathfrak{m}^s\subset\mathfrak{c}$. Since the radical of $\mathfrak{m}B$ is J, there is a positive integer t such that $J^{st}\subset\mathfrak{m}^sB\subset\mathfrak{c}B=\mathfrak{c}\subset\mathfrak{m}$. Then the implication (i)⇒(ii) is proved in the exactly same way as the implication (1)⇒(2) of Proposition 2.2 in [5]. Therefore we omit the detail.

PROPOSITION 2. *Let* B *be a semilocal ring with the maximal ideals* $\mathfrak{n}_1,\dots,\mathfrak{n}_n$, *and* A *a noetherian local subring of* B *with maimal ideal* $\mathfrak{m}$. *Assume that* B *is a finite* A*-module and the conductor* $A:_A B$ *of* A *in* B *is* $\mathfrak{m}$*-primary. If* $G(A)$ *is reduced and we have*

$$\operatorname{emdim}(A) = \textstyle\sum_{i=1}^{n} [\kappa(\mathfrak{n}_i):\kappa(\mathfrak{m})]\operatorname{emdim}(B_{\mathfrak{n}_i}),$$

then A *is seminormal in* B. *Moreover if* $G(B)$ *is reduced, then the converse is also true.*

PROOF. If $G(A)$ is reduced, then the natural homomorphism $G(A)\to G(B)$ is injective by Proposition 1. In particular we see that $J^2\cap\mathfrak{m}=\mathfrak{m}^2$ with $J=\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$, and hence our first assertion is a direct consequence of Proposition 3 in [7]. Conversely if A is seminormal in B, then $G(A)$ is isomorphic to a subring of $G(B)$ and the equality in our proposition is true by Lemma 4 in [7]. Therefore if $G(B)$ is reduced, then so is $G(A)$.

Next we need the following lemmas to prove Theorem 1.

LEMMA 1. *Let* B *be a noetherian semilocal domain with the maximal ideals* $\mathfrak{n}_1,\dots,\mathfrak{n}_n$, *and* A *a local subring of* B *with maximal ideal* $\mathfrak{m}$ *such that the quotient field of* A *is equal to that of* B. *Assume that* B *is a finite* A*-module and that* $\dim A=\dim B_{\mathfrak{n}_i}$ *for each* $i=1,\dots,n$. *If we have* $e(A)=\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i):\kappa(\mathfrak{m})]$, *then we have* $e(\mathfrak{m}B_{\mathfrak{n}_i})=1$ *for each* $i=1,\dots,n$.

PROOF. This is a direct consequence of Corollary 1 to Theorem 24 of Chapter VIII in [8].

LEMMA 2. *Let* A *be a noetherian reduced local ring with maximal ideal* $\mathfrak{m}$, *and let* $\bar{A}$ *be the integral closure of* A *in the total quotient ring* $Q(A)$ *of* A. *Assume that* $\bar{A}$ *is a finite* A*-module and let* $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ *be the maximal ideals of* $\bar{A}$. *If we have* $e(A)=\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i):\kappa(\mathfrak{m})]$ *and* $\dim A=\dim\bar{A}_{\mathfrak{n}_i}$ *for each* $i=1,\dots,n$, *then the multiplicity* $e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i})$ *is equal to* 1 *for each* $i=1,\dots,n$.

PROOF. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime divisors of A and K_i the quotint field $Q(A/\mathfrak{p}_i)$ for each $i=1, \dots, s$. If B_i is the integral closure of $A/\mathfrak{p}_i$ in K_i and we identify $Q(A)$ with the direct product $K_1 \times \cdots \times K_s$, then $\bar{A}$ may be considered as the direct product $B_1 \times \cdots \times B_s$. If we denote by $\bar{\mathfrak{p}}_1$ the minimal prime ideal $(0) \times B_2 \times \cdots \times B_s$ of $\bar{A}$, B_1 is isomorphic to $\bar{A}/\bar{\mathfrak{p}}_1$ and any maximal ideal $\mathfrak{n}$ of $\bar{A}$ containing $\bar{\mathfrak{n}}_1$ is the form $\bar{\mathfrak{n}} \times B_2 \times \cdots \times \bar{B}_s$ where $\bar{\mathfrak{n}}$ is a maximal ideal of B_1. Then we see that $\bar{A}_{\bar{\mathfrak{n}}} = (B_1)_{\bar{\mathfrak{n}}}$, and hence it follows from our assumption that $\dim A/\mathfrak{p}_1 = \dim B_1 = \dim (B_1)_{\bar{\mathfrak{n}}} = \dim \bar{A}_{\mathfrak{n}} = \dim A$. Similarly we see that $\dim A/\mathfrak{p}_i = \dim A$ for each $i=2, \dots, s$. Therefore the additive theorem on multiplicity (cf. (23.5) in [4]) implies the following equality

$$e(A) = e(\mathfrak{m}) = \textstyle\sum_{i=1}^{s} e(\mathfrak{m}/\mathfrak{p}_i) = \sum_{i=1}^{s} e(A/\mathfrak{p}_i) \tag{$*$}$$

and we see also from Corollary 1 to Theorem 24 of Chapter VIII in [8]

$$e(A/\mathfrak{p}_i) = \textstyle\sum_{j=1}^{n_j} [\kappa(\bar{\mathfrak{n}}_{ij}) : \kappa(\mathfrak{m})] e(\mathfrak{m}(B_i)_{\bar{\mathfrak{n}}_{ij}}) \tag{$**$}$$

where $\bar{\mathfrak{n}}_{ij}$ $(j=1, \dots, n_i)$ are the maximal ideals of B_i. If $\mathfrak{n}_{ij}$ is the maximal ideal of $\bar{A}$ naturally corresponding to $\bar{\mathfrak{n}}_{ij}$ for $j=1, \dots, n_i$, we see $(B_i)_{\bar{\mathfrak{n}}_{ij}} = \bar{A}_{\mathfrak{n}_{ij}}$ and $\kappa(\mathfrak{n}_{ij}) = \kappa(\bar{\mathfrak{n}}_{ij})$. Therefore we see

$$\begin{aligned} e(A/\mathfrak{p}_i) &= \textstyle\sum_{j=1}^{n_i} [\kappa(\bar{\mathfrak{n}}_{ij}) : \kappa(\mathfrak{m})] e(\mathfrak{m}(B_i)_{\bar{\mathfrak{n}}_{ij}}) \\ &\geqq \textstyle\sum_{j=1}^{n_i} [\kappa(\mathfrak{n}_{ij}) : \kappa(\mathfrak{m})] \end{aligned}$$

and hence it follows from $e(A) = \sum_{i=1}^{n} [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})]$ that $e(A/\mathfrak{p}_i) = \sum_{j=1}^{n_i} [\kappa(\bar{\mathfrak{n}}_{ij}) : \kappa(\mathfrak{m})]$ for $i=1, \dots, s$, because we have $\{\mathfrak{n}_{ij} \mid 1 \leqq i \leqq s, 1 \leqq j \leqq n_i\} = \{\mathfrak{n}_1, \dots, \mathfrak{n}_n\}$. By Lemma 1 this means that $e(\mathfrak{m}(B_i)_{\bar{\mathfrak{n}}_{ij}}) = 1$ for any i, j with $1 \leqq i \leqq s$ and $1 \leqq j \leqq n_i$. Therefore we see $e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i}) = 1$ for each $i=1, \dots, n$.

The equalities $(*)$ and $(**)$ in the proof of Lemma 2 shows the following

COROLLARY. *Let A, $\bar{A}$, $\mathfrak{m}$, $\mathfrak{n}_1, \dots, \mathfrak{n}_n$ be as above. If* $\dim A = \dim \bar{A}_{\mathfrak{n}_i}$ *for each $i=1, \dots, n$, then we have*

$$e(A) = \textstyle\sum_{i=1}^{n} [\kappa(\mathfrak{n}_i) : \kappa(\mathfrak{m})] e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i}).$$

THEOREM 1. *Let A be a noetherian reduced local ring with maximal ideal $\mathfrak{m}$ and $\bar{A}$ the integral closure of A in its total quotient ring $Q(A)$. Assume that $\bar{A}$ is a finite A-module and regular, that* $\dim A = \dim \bar{A}_{\mathfrak{n}}$ *for any maximal ideal $\mathfrak{n}$ of $\bar{A}$ and that the conductor $A :_A \bar{A}$ of A on $\bar{A}$ is $\mathfrak{m}$-primary. Then the following are equivalent:*

(i) *A is seminormal.*

(ii) *$G(A)$ is reduced and* $\operatorname{emdim}(A) = e(A) \times \dim A$.

PROOF. If A is seminormal, then $G(A)$ is isomorphic to a graded subring of $G(\bar{A}_{\mathfrak{n}_1})\oplus\cdots\oplus G(\bar{A}_{\mathfrak{n}_n})$ and we have

$$\operatorname{emdim}(A) = \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]\operatorname{emdim}(\bar{A}_{\mathfrak{n}_i})$$

by Proposition 3 and Lemma 4 in [7] where $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ are the maximal ideals of $\bar{A}$. Since $\bar{A}$ is regular, each $\bar{A}_{\mathfrak{n}_i}$ is a regular local ring and hence $G(\bar{A}_{\mathfrak{n}_i})$ is isomophic to a polynomial ring over $\kappa(\mathfrak{n}_i)$. In particular $G(\bar{A}_{\mathfrak{n}_i})$ is reduced and hence so is $G(A)$. We see also

$$\operatorname{emdim}(\bar{A}_{\mathfrak{n}_i}) = \dim \bar{A}_{\mathfrak{n}_i} = \dim A.$$

On the other hand we see by Proposition 3 in [7]

$$\mathfrak{m}\bar{A}_{\mathfrak{n}_i} = (\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n)\bar{A}_{\mathfrak{n}_i} = \mathfrak{n}_i\bar{A}_{\mathfrak{n}_i}.$$

Therefore it follows from Corollary to Lemma 2 and regularity of $\bar{A}$ that

$$\begin{aligned} e(A) = e(\mathfrak{m}) &= \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i}) \\ &= \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]e(\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i}) \\ &= \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]. \end{aligned}$$

This means that we have $\operatorname{emdim}(A)=e(A)\times\dim A$.

Conversely if $G(A)$ is reduced, then the natural homomorphism $G(A)\to G(\bar{A})$ is injective by Proposition 1. In particular we see that the natural mapping

$$f\colon \mathfrak{m}/\mathfrak{m}^2 \longrightarrow J/J^2=\mathfrak{n}_1/\mathfrak{n}_1^2\oplus\cdots\oplus\mathfrak{n}_n/\mathfrak{n}_n^2$$

is injective, where J is the Jacobson radical $\mathfrak{n}_1\cap\cdots\cap\mathfrak{n}_n$ of $\bar{A}$. Therefore we see by Corollary to Lemma 2 and regularity of $\bar{A}$ the following inequality

$$\begin{aligned} \operatorname{emdim}(A) &\leqq \textstyle\sum_{i=1}^{n}\dim_{\kappa(\mathfrak{m})}\mathfrak{n}_i/\mathfrak{n}_i^2 \\ &= \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]\operatorname{emdim}(\bar{A}_{\mathfrak{n}_i}) \\ &= \textstyle\sum_{i=1}^{n}[\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]\dim(\bar{A}_{\mathfrak{n}_i}) \\ &\leqq e(A)\times\dim(A). \end{aligned}$$

If the equality of (ii) in our theorem is true, then we have $\operatorname{emdim}(A)=\sum_{i=1}^{n}\dim\kappa_{(\mathfrak{m})}\mathfrak{n}_i/\mathfrak{n}_i^2$ from the above inequality and hence f is bijective. This means by Proposition 3 in [7] that A is seminormal.

REMARK. If $\dim A=1$, then Theorem 1 is the main theorem of [2]. In the paper [1] C. Cumino also obtained some results closely related to Theorem 1 for two dimensional cases.

Now we give a geometric interpretation of the above theorem. For this purpose we recall some algebro-geometric terminologies. Let k be an algebraically closed field of any characteristic and V a reduced algebraic scheme over k. Let A be the stalk of the structure sheaf of V at a closed point P. Then A is a reduced local ring. If $\mathfrak{m}$ is the maximal ideal of A, then the residue field $A/\mathfrak{m}$ is isomorphic to k. The dual space $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ of the vector space $\mathfrak{m}/\mathfrak{m}^2$ over k is called the *Zariski tangent space* of V at P and denoted by $T_P(V)$. Moreover the affine scheme $\mathrm{Spec}\,(G(A))$ is called the *tangent cone* of V at P. It is easy to see that if we identify $T_P(V)$ with the algebraic sheme $\mathrm{Spec}\,(k[t_1,\dots, t_r])$ where $r=\dim_k \mathfrak{m}/\mathfrak{m}^2$, then $\mathrm{Spec}\,(G(A))$ is naturally considered as a closed subscheme of $T_P(V)$. Since $G(A)$ is a graded k-algebra, we can consider $\mathrm{Proj}\,(G(A))$, which we call the *projectivized tangent cone* of V at P. If $\bar{A}$ is the integral closure of A in its total quotient ring $Q(A)$, then $\bar{A}$ is a semilocal ring with a finite number of maximal ideals $\mathfrak{n}_1,\dots, \mathfrak{n}_n$ and we say that the *branch number* of V at P is n. Moreover we say that *V has a non-singular normalization near P*, if $\bar{A}$ is regular and that P is a *seminormal point* of *V if A* is seminormal.

To show Theorem 2 we need the following

LEMMA 3. *Let k, V, P, A, $\mathfrak{m}$ and $\bar{A}$ be as above, and assume that V is equidimensional. Then V has a non-singular normalization near P and $\mathfrak{m}\bar{A}$ is the Jacobson radical of $\bar{A}$ if and only if $e(A)$ is equal to the branch number of V at P.*

PROOF. If $\mathfrak{p}$ is any minimal prime ideal of A, then we have $\dim A=\dim A/p$ by the equidimensionality of V. Moreover if B is integral closure of $A/\mathfrak{p}$ in its total quotient field $Q(A/\mathfrak{p})$, then it is well known that $\dim A/\mathfrak{p}=\dim B_{\mathfrak{n}}$ for any maximal ideal $\mathfrak{n}$ of B. Therefore we see easily that $\dim A=\dim \bar{A}_{\mathfrak{n}_i}$ for any $\mathfrak{n}_i$, where $\mathfrak{n}_1,\dots, \mathfrak{n}_n$ are the maximal ideals of $\bar{A}$. By Corollary to Lemma 2 we have

$$e(A) = \textstyle\sum_{i=1}^{n} [\kappa(\mathfrak{n}_i)\colon \kappa(\mathfrak{m})]e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i}).$$

This equality means that $e(A)$ is equal to the brach number n of V at P if and only if $e(\mathfrak{m}\bar{A}_{n_i})=1$ for each $i=1,\dots, n$. If $\bar{A}$ is regular and $\mathfrak{m}\bar{A}$ is the Jacobson radical of $\bar{A}$, then we see $e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i})=e(\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i})=e(\bar{A}_{\mathfrak{n}_i})=1$ for each $i=1,\dots, n$. Conversely if we have $e(\mathfrak{m}\bar{A}_{\mathfrak{n}_i})=1$ for each $i=1,\dots, n$, then we see easily $e(\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i})=1$. Since $\bar{A}_{\mathfrak{n}_i}$ is a localization of an affine algebra over a field k, $\bar{A}_{\mathfrak{n}_i}$ is *unmixed* in the sense of §25 in [4]. Therefore $\bar{A}_{\mathfrak{n}_i}$ is a regular local ring by (40.6) in [4]. Moreover we can see easily $\mathfrak{m}\bar{A}_{\mathfrak{n}_i}=\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i}$ for each i in the same way in the proof of Proposition 4 in [7], but we omit the detail.

THEOREM 2. *Let V be a reduced equidimensional algebraic scheme over an algebraically closed field k and let A be the stalk of the structure sheaf of V at a closed point P. Assume that P is an isolated singularity of V. Then P is a* *seminormal point of V and V has a non-singular normalization near P if and only if the following are satisfied*:

(i) *The multiplicity $e(A)$ of A is equal to the branch number of V at P.*

(ii) $\dim_k T_P(V) = e(A) \times \dim V$.

(iii) *The tangent cone* $\operatorname{Spec}(G(A))$ *of V at P is reduced.*

PROOF. Let $\mathfrak{m}$ be the maximal ideal of A and let $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ be the maximal ideals of the integral closure $\bar{A}$ of A in its total quotient ring $Q(A)$. Since P is an isolated singularity of V, the conductor $A:{}_A\bar{A}$ of A in $\bar{A}$ is $\mathfrak{m}$-primary. Moreover we have $\dim A = \dim \bar{A}_{\mathfrak{n}_i}$ for each $i = 1,\dots, n$ as seen in the proof of Lemma 3. If the assertion (i) is true, then $\bar{A}$ is regular by Lemma 3. Therefore if the assertions (i), (ii) and (iii) are true, then A is seminormal and $\bar{A}$ is regular by Theorem 1.

Conversely if A is seminormal and $\bar{A}$ is regular, then the assertion (ii) and (iii) are true by Theorem 1 and the Jacobson radical $\mathfrak{n}_1 \cap \cdots \cap \mathfrak{n}_n$ of $\bar{A}$ is equal to $\mathfrak{m}$ by Proposition 3 in [7]. Therefore the assertion (i) is also true by Lemma 3.

2. Generalized multicloss singularities

In this section we give a definition of generalized multicross singularities of an algebraic scheme over an algebraically closed field k and show some basic properties of these points. Let P be a closed point of a reduced algebraic scheme over k. Let A, $\mathfrak{m}$, $\bar{A}$ and $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ be as before. Let φ_i be the natural linear mapping of $T_i = \operatorname{Hom}_k(\mathfrak{n}_i/\mathfrak{n}_i^2, k)$ to $T_P(V) = \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ defined dually from the natural mapping $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}_i/\mathfrak{n}_i^2$, and put $\bar{T}_i = \varphi_i(T_i)$. Then we say that P is *an n-fold generalized multicross point* of V, if the following are satisfied:

(i) *The multiplicity $e(A)$ of A is equal to the branch number n of V at P.*

(ii) *For any $i \neq j$, we have $\bar{T}_i \cap \bar{T}_j = \{0\}$.*

It is easy to see that a generalized multicross point of an algebraic reduced curve is nothing else than an ordinary singular point in the usual sense.

PROPOSITION 3. *Let V, P, A, φ_i, be as above. If V is equidimensional and P is an n-fold generalized multicross singularity of V, then the following are true*:

(i) *V has a non-singular normalization near P.*

(ii) *φ_i is injective for each $i = 1,\dots, n$.*

(iii) *If P is not a simple point of V, then the dimension of the Zariski tangent space $T_P(V)$ of V at P is not less than* $2\dim V$.

PROOF. The assertion (i) follows from Lemma 3. Now φ_i is injective if and only if the natural mapping $\mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}_i/\mathfrak{n}_i^2$ is surjective, that is, $\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i}$ is equal to $\mathfrak{m}\bar{A}_{\mathfrak{n}_i} + \mathfrak{n}_i^2\bar{A}_{\mathfrak{n}_i}$. Therefore, by Nakayama lemma, φ_i is injective if and only if $\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i} = \mathfrak{m}\bar{A}_{\mathfrak{n}_i}$. However it follows from Lemma 3 and $e(A) = n$ that $\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i} = \mathfrak{m}\bar{A}_{\mathfrak{n}_i}$ for each $i=1,\dots, n$. The last assertion is a direct consequence of (i), (ii) and the equality

$$\dim V = \dim A = \dim \bar{A} = \dim \bar{A}_{\mathfrak{n}_i} = \operatorname{emdim}(\bar{A}_{\kappa_i}) = \dim T_i.$$

Remark. Let V_0 be an affine algebraic variety of dimension r over an algebraically closed field k, and let $P_1,\dots, P_n$ be simple points of V_0. Then we know by Example (1) of §2 in [7] that there exists an algebraic variety V' and a closed seminormal point P' of V' which is obtained from V_0 by glueing n points $P_1,\dots, P_n$. If A' is the stalk of the structure sheaf of V' at P', then it is easy to see that by Lemma 4 in [7] that the tangent cone $\operatorname{Spec}(G(A'))$ of V' at P' consists of n linear spaces $T_1,\dots, T_n$ of dimension r and that the Zariski tangent space $T_{P'}(V')$ of V' at P' is isomorphic to the direct sum $T_1\oplus\cdots\oplus T_n$ as vector spaces over k. If π: $V'\to V$ is a generic projection of V' to an algebraic variety V contained in an affine space of dimension $2r$, then $P=\pi(P')$ is an n-fold generalized multicross singularity of V such that $\dim T_P(V)=2r=2\times\dim V$. The proof is routine and not so difficult, but we omit the detail.

Next we shall give a criterion in terms of projectivized tangent cones for an isolated singularity of an algebraic scheme to be a generalized multicross point. For this purpose we need the following lemma which is known in the case of curves (cf. Lemma 2.2 in [5]).

Lemma 4. *Let A be a reduced noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\bar{A}$ be the integral closure of A in its total quotient ring $Q(A)$. Assume that the conductor $\mathfrak{c}=A :_A \bar{A}$ of A in $\bar{A}$ is $\mathfrak{m}$-primary and that $G(\bar{A})$ is reduced. If the Jacobson radical J of $\bar{A}$ is equal to $\mathfrak{m}\bar{A}$, then the kernel of the natural homomorphism f: $G(A)\to G(\bar{A})$ coincides with the nilradical of $G(A)$.*

Proof. It is clear from the reducedness of $G(\bar{A})$ that any nilpotent element of $G(A)$ is contained in the kernel of f. Since $\mathfrak{c}$ is $\mathfrak{m}$-primary, there is a positive integer $h>0$ with $\mathfrak{c}\supset\mathfrak{m}^h$ and then we see $\mathfrak{m}^n\bar{A}\subset\mathfrak{c}\bar{A}\subset\mathfrak{m}$ for $n\geq h$. Let x be an element of $\mathfrak{m}^i\setminus\mathfrak{m}^{i+1}$ such that the leading form $\bar{x}\in\mathfrak{m}^i/\mathfrak{m}^{i+1}$ of x is contained in the kernel of f. Then we see $x\in J^{i+1}$ and hence $x^n\in J^{n(i+1)}=(\mathfrak{m}\bar{A})^{n(i+1)}=\mathfrak{m}^{n(i+1)}\bar{A}$. Therefore if $n\geq h$, then we see that

$$x^n\in\mathfrak{m}^{n(i+1)}\bar{A} = \mathfrak{m}^{ni}\mathfrak{m}^n\bar{A} \subset \mathfrak{m}^{ni}\mathfrak{m} = \mathfrak{m}^{ni+1}.$$

This means that $\bar{x}^n=0$. Since the kernel of f is a homogeneous ideal of $G(A)$, our assertion is proved.

Corollary. *Let V, P, A and $\mathfrak{m}$ be as in Proposition 3. Assume that V is equidimensional, that P is an isolated singularity of V and that the multiplicity $e(A)$ of A is equal to the branch number of V at P. Then any irreducible* *component of the tangent cone* Spec$(G(A))$ *of* V *at* P *is the set-theoretical image of some irreducible component of* Spec$(G(\bar{A}))$, *where* $\bar{A}$ *is the integral closure of* A *in its total quotient ring* $Q(A)$.

PROOF. By Lemma 3 $\bar{A}$ is regular and $\mathfrak{m}\bar{A}$ is equal to the Jacobson radical of $\bar{A}$. If $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ are the maximal ideals of $\bar{A}$, then $\mathfrak{m}\bar{A}_{\mathfrak{n}_i}=\mathfrak{n}_i\bar{A}_{\mathfrak{n}_i}$ for $i=1,\dots,n$. Let f be the natural homomorphism of $G(A)$ to $G(\bar{A})=G(\bar{A}_{\mathfrak{n}_1})\oplus\cdots\oplus G(\bar{A}_{\mathfrak{n}_n})$ and ρ_i the projection of $G(\bar{A})$ to $G(\bar{A}_{\mathfrak{n}_i})$. Then the comosite $\rho_i\circ f$ is a surjective homomorphism of $G(A)$ to $G(\bar{A}_{\mathfrak{n}_i})$ and hence $G(\bar{A})$ is integral over $f(G(A))$. Since $G(\bar{A}_{\mathfrak{n}_i})$ is a polynomial ring over k for each $i=1,\dots,n$, we see by Lemma 4 that $G(A)/\mathfrak{a}$ is isomorphic to $f(G(A))$ where $\mathfrak{a}$ is the nilradical of $G(A)$. If $\mathfrak{p}$ is any minimal prime ideal of $G(A)$, then $\mathfrak{p}$ contains $\mathfrak{a}$ and there exists a prime ideal $\mathfrak{q}$ of $G(\bar{A})$ by the lying-over theorem such that $f^{-1}(\mathfrak{q})=\mathfrak{p}$. This prime ideal $\mathfrak{q}$ must be minimal, and hence our assertion is proved.

THEOREM 3. *Let* V, P *and* A *be as above. Assume that* V *is equidimensional and that* P *is an isolated singularity of* V. *Then* P *is an* n*-fold generalized multicross point of* V *if and only if the following are satisfied*:

(i) *The multiplicity* $n=e(A)$ *is equal to the branch number of* V *at* P.

(ii) *The projectivized tangent cone* Proj$(G(A))$ *of* V *at* P *has* n *connected components.*

PROOF. If P is an n-fold generalized multicross point of V, then the assertion (i) is true by definition. If $\bar{A}$ is the normalization of $\bar{A}$ in $Q(A)$, then $G(\bar{A})$ is identified with $G(\bar{A}_{\mathfrak{n}_1})\oplus\cdots\oplus G(\bar{A}_{\mathfrak{n}_n})$ where $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ are the maximal ideals of $\bar{A}$. Therefore the minimal prime ideals of $G(\bar{A})$ are $\mathfrak{q}_1=(0)\oplus G(\bar{A}_{\mathfrak{n}_2})\oplus\cdots G(\bar{A}_{\mathfrak{n}_n}),\dots,$ $\mathfrak{q}_n=G(\bar{A}_{\mathfrak{n}_1})\oplus\cdots\oplus G(\bar{A}_{\mathfrak{n}_{n-1}})\oplus(0)$. Let f and f^* be the natural homomorphisms of $G(A)$ and its cohomomorphism of Spec$(G(\bar{A}))$ to Spec$(G(A))$, respectively. Since $\bar{A}_{\mathfrak{n}_i}$ is a regular local ring for each $i=1,\dots,n$, Spec$(G(\bar{A}_{\mathfrak{n}_i}))$ is a linear variety and hence we see easily that the image $f^*($Spec$(G(\bar{A}_{\mathfrak{n}_i}))$ is equal to $\bar{T}_i$ in $T_P(V)$ set-theoretically. Since, by Corollary to Lemma 4, any irreducible component of Spec$(G(A))$ is the image of that of Spec$(G(\bar{A}))$ under f^* and $\bar{T}_i\cap\bar{T}_j=\{0\}$ in $T_P(V)$ for $i\neq j$, we see that the number of irreducible components of Spec$(G(A))$ is n and hence that the number of connected components of Proj$(G(A))$ is n.

Conversely assume that the assertions (i) and (ii) are true. Then the number of minimal prime ideals of $G(\bar{A})$ is n and any minimal prime ideal $\mathfrak{p}$ of $G(A)$ is the inverse image $f^{-1}(\mathfrak{q})$ of a minimal prime ideal $\mathfrak{q}$ of $G(\bar{A})$ by Corollary to Lemma 4. Therefore (ii) implies that $f^*($Spec$(G(\bar{A}_{\mathfrak{n}_i}))$ and $f^*($Spec$(G(\bar{A}_{\mathfrak{n}_j}))$ are different from each other for $i\neq j$, and hence we see that $\bar{T}_i\cap\bar{T}_j=\{0\}$ for $i\neq j$.

The following theorem shows that any isolated seminormal singularity with a non-singular normalization near it is a generalized multicross point.

Theorem 4. *Let V, P, $T_P(V)$ and $\overline{T}_i$ be as before and assume that V is equidimensional. Then the following are equivalent:*

(i) *P is an n-fold generalized multicross point of V and $T_P(V)=\overline{T}_1\oplus\cdots\oplus\overline{T}_n$.*

(ii) *P is an isolated seminormal singularity of V and V has a non-singular normalization near P.*

Proof. Let A be the stalk of the structure sheaf of V at P as before and $\mathfrak{m}$ the maximal ideal of A. Let $\bar{A}$ be the normalization of A in $Q(A)$, and let $\mathfrak{n}_1,\dots,\mathfrak{n}_n$ be the maximal ideals of $\bar{A}$. If (i) is true, then V has the non-singular normalization near P and φ_i: $T_i\to\overline{T}_i=\varphi_i(T_i)$ is injective for $i=1,\dots,n$ by Proposition 3. Therefore $T_P(V)=\overline{T}_1\oplus\cdots\oplus\overline{T}_n=\varphi_1(T_1)\oplus\cdots\oplus\varphi_n(T_n)$ implies that the linear mapping: $T_1\oplus\cdots\oplus T_n\to T_P(V)$ induced from the natural mapping: $\mathfrak{m}/\mathfrak{m}^2\to J/J^2=\mathfrak{n}_1/\mathfrak{n}_1^2\oplus\cdots\oplus\mathfrak{n}_n/\mathfrak{n}_n^2$ is injective and hence $\mathfrak{m}/\mathfrak{m}^2\to J/J^2$ is surjective where J is the Jacobson radical of $\bar{A}$. This means that A is seminormal and the conductor $A:_A\bar{A}$ of A in $\bar{A}$ is $\mathfrak{m}$-primary by Proposition 3 in [7].

Conversely if (ii) is true, then we see $J=\mathfrak{m}$ by Proposition 3 in [7]. Therefore $\mathfrak{m}/\mathfrak{m}^2$ is isomorphic to $J/J^2\cong\mathfrak{n}_1/\mathfrak{n}_1^2\oplus\cdots\oplus\mathfrak{n}_n/\mathfrak{n}_n^2$ and hence $T_P(V)=\varphi_1(T_1)\oplus\cdots\oplus\varphi_n(T_n)=\overline{T}_1\oplus\cdots\oplus\overline{T}_n$. We see also that $e(A)$ is equal to the branch number of V at P by Lemma 3. Therefore the assertion (i) is true.

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