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# ANTI-INTEGRAL EXTENSIONS OF NOETHERIAN DOMAINS

*Dedicated to Professor Masayoshi Nagata on his 60th birthday*

By Susumu ODA and Ken-ichi YOSHIDA

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**INTRODUCTION.** In [1, Ex. 4, page 12], it is mentioned that if $\alpha$ is a non-zero element in the quotient field $K$ of an integral domain $R$, then $R(\alpha) = R[\alpha] \cap R[1/\alpha]$ is integral over $R$. The ring $R(\alpha)$ has seemed rather interesting to us, but we do not recall seeing any other results in the literature about such rings except a few ones (eg., [3]).

Let $R$ be a Noetherian domain and $K$ its quotient field. Let $\alpha$ be a non-zero element in $K$ and $R(\alpha) = R[\alpha] \cap R[1/\alpha]$. We call $\alpha$ anti-integral over $R$ if $R(\alpha) = R$. We investigate a simple extension $R[\alpha]$ of $R$ with $\alpha$ anti-integral over $R$, which is called an anti-integral extension of $R$.

Recall the following fact:

Let $A$ be an over-ring of $R$ (i.e., $R \subset A \subset K$) which is of finite type over $R$. Assume that $A$ is quasi-finite over $R$ and $R$ is normal. Then if a contraction map: $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ is surjective, we have $R = A$ (cf. [6]).

This is known as the Zariski-Main Theorem (ZMT). If $R$ is not integrally closed, we can find a counter-example. Even if $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ is surjective, $A$ is not necessarily integral over $R$ (in particular, $R \neq A$). But we can show that if $A$ is $R[\alpha_1, \ldots, \alpha_n]$, each $\alpha_i$ anti-integral over $R$, then the surjectivity of $\mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ yields $A = R$. This result is our first motive for writing this paper (See Section 2).

Section 1 is devoted to establishing several properties of an anti-integral extension $R[\alpha]$ of $R$. For example:

(a) If $\alpha$ is integral over $R$ then $R(\alpha) = R[\alpha]$.

(b) When $\alpha$ is anti-integral over a local domain $(R, m)$ and $R \subsetneq R[\alpha]$, $mR[\alpha] = R[\alpha]$ if and only if $R = R(\alpha) = R[1/\alpha] \longrightarrow R[\alpha, 1/\alpha]$ is a flat extension.

In Section 2, we show mainly:

(c) Whenever $\alpha$ is anti-integral over $R$, $R[\alpha]$ is flat over $R$ if and only if $I_\alpha + \alpha I_\alpha = R$, where $I_\alpha$ denotes an ideal $(R :_R \alpha)$ of $R$.

(d) Let $A = R[\alpha_1, \ldots, \alpha_n]$ with each $\alpha_i$ anti-integral over $R$. Assume that $R$

contains an infinite field $k$ and either that $R$ satisfies the altitude formula or that $A$ is quasi-finite over $R$ (cf. [6]). Then if the contraction map $\phi$: $\operatorname{Spec} A \longrightarrow \operatorname{Spec} R$ satisfies $\operatorname{Im} \phi \supset \operatorname{Dp}_1(R) := \{ p \in \operatorname{Spec} R \mid \operatorname{depth} R_p = 1 \}$, then $A = R$.

In Section 3, we introduce a ring $R(J)$ for a fractional ideal $J$ of $R$ and show:
(e) If $R(I_\alpha) = R(\alpha \in K)$, then $\alpha$ is anti-integral over $R$.

In this paper all rings are assumed to be commutative and to have an identity, and our general references for unexplained technical terms are [2] and [4]. Throughout this paper we fix the notation as follows (unless otherwise specified): Let $R$ denote a Noetherian domain, $K$ the quotient field of $R$ and $\bar{R}$ the integral closure of $R$ in $K$. For $\alpha \in K$, $I_\alpha$ denotes an ideal $(R :_R \alpha) = \{ a \in R \mid a\alpha \in R \}$ of $R$.

**1. The ring $R(\alpha)$.**
We start with the following definition.

DEFINITION 1.1. For a non-zero element $\alpha$ of $K$, we set $R(\alpha) = R[\alpha] \cap R[1/\alpha]$ in $K$.

The following lemma is seen in [1, Ex. 4], but we prove this for the sake of completeness.

LEMMA 1.2. *For a non-zero $\alpha \in K$, $R(\alpha)$ is integral over $R$.*

PROOF. We know that $\bar{R}$ can be expressed as $\cap V_\lambda$, where $V_\lambda$'s are discrete valuation over-rings containing $R$. Since $V_\lambda$ is a valuation ring, we have that $\alpha \in V_\lambda$ or $1/\alpha \in V_\lambda$. So $R(\alpha) = R[\alpha] \cap R[1/\alpha] \subset V_\lambda$ for each $V_\lambda$. Hence $R(\alpha) \subset \bar{R}$.
Q.E.D.

The following elementary lemma is useful in the sequel.

LEMMA 1.3. *For a non-zero $\alpha \in K$, the following statements are equivalent:*
(1) *$\alpha$ is integral over $R$.*
(2) $R(\alpha) = R[\alpha]$.
(3) $\alpha \in R(\alpha)$.

PROOF. (2) $\Leftrightarrow$ (3) is obvious. (2) $\Rightarrow$ (1) follows from Lemma 1.2. (1) $\Rightarrow$ (2): Let $\alpha$ be integral over $R$. Then

$$\alpha^n + a_1 \alpha^{n-1} + \cdots + a_n = 0 \qquad (a_i \in R).$$

So $\alpha = -(a_1 + a_2(1/\alpha) + \cdots + a_n(1/\alpha)^{n-1}) \in R[1/\alpha]$, and hence $R[\alpha] \subset R[1/\alpha]$. Thus $R[\alpha] \subset R[\alpha] \cap R[1/\alpha] = R(\alpha)$. Q.E.D.

DEFINITION 1.4. Assume that $R = R(\alpha)$ for a non-zero $\alpha \in K$. Then we say that $\alpha$ is an anti-integral element or that $\alpha$ is anti-integral over $R$. When $\alpha$ is anti-integral over $R$, a simple extension $R[\alpha]$ is said to be an anti-integral extension of $R$.

The property of being anti-integral is local-global as is seen in the next lemma.

LEMMA 1.5. *For a non-zero* $\alpha$ *in* $K$, *the following statements are equivalent:*
(1) $R(\alpha) = R$.
(2) $R_p(\alpha) = R_p$ *for each* $p \in \mathit{Spec}\, R$.
(3) $R_p(\alpha) = R_p$ *for each* $p \in Dp_1(R) = \{p \in \mathit{Spec}\, R \mid \mathit{depth}\, R_p = 1\}$.

PROOF. (1) $\Rightarrow$ (2): Take $\beta \in R_p(\alpha)$. Then

$$\beta = a_n \alpha^n + \cdots + a_0 = b_m (1/\alpha)^m + \cdots + b_0$$

with $a_i, b_j \in R_p$. Choose a common denominator $d \in R$ of $a_i$'s and $b_j$'s such that $d \notin p$. Then $d\beta \in R(\alpha) = R$, and hence $\beta \in R_p$. (2) $\Rightarrow$ (3) is obvious. (3) $\Rightarrow$ (1): Take $\beta \in R(\alpha)$. Suppose $\beta \notin R$. Then $I_\beta \neq R$. Let $p$ be a prime divisor of $I_\beta$. Then depth $R_p = 1$ (cf. [8]). Hence $\beta \in R_p(\alpha) = R_p$, which implies that $I_\beta \not\subset p$, contradiction. Q.E.D.

REMARK 1.6. (1) If $\alpha \in K$ is anti-integral over $R$, $\alpha$ is also anti-integral over $R_p$ for each $p \in \operatorname{Spec} R$.
(2) For a non-zero $\alpha \in K$, $\alpha$ is anti-integral and integral over $R$ if and only if $\alpha \in R$.

PROPOSITION 1.7. *For a non-zero* $\alpha \in K$, *the following statements are equivalent:*
(1) $R[\alpha] \cap \bar{R} = R$ (*i.e., $R$ is integrally closed in* $R[\alpha]$).
(2) *For any non-zero* $\beta \in R[\alpha]$, $R(\beta) = R$ (*i.e.,* $\beta \in R[\alpha]$ *is anti-integral over* $R$).

PROOF. (2) $\Rightarrow$ (1): Suppose that $R[\alpha] \cap \bar{R} \supsetneqq R$. Then there exists $\beta \in R[\alpha] \cap \bar{R}$ but $\beta \notin R$. Being $\beta \in \bar{R}$ implies that $R(\beta) = R[\beta] \supsetneqq R$ by Lemma 1.3. Thus $R(\beta) \supsetneqq R$. (1) $\Rightarrow$ (2): Suppose that $R[\alpha] \cap \bar{R} = R$. Since $R \subset R(\alpha) \subset \bar{R} \cap R[\alpha] = R$, we have $R(\alpha) = R$. For any non-zero $\beta \in R[\alpha]$, we have $R \subset R[\beta] \cap \bar{R} \subset R[\alpha] \cap \bar{R} = R$. So $R[\beta] \cap \bar{R} = R$ and $R(\beta) = R$. Q.E.D.

REMARK 1.8. For each non-zero $\alpha \in K$, we have $R(\alpha)(\alpha) = R(\alpha)$, i.e., $\alpha$ is anti-integral over $R(\alpha)$. In other words, $R[\alpha]$ is an anti-integral extension of $R(\alpha)$. In fact, this is implied by: $R(\alpha)(\alpha) = R(\alpha)[\alpha] \cap R(\alpha)[1/\alpha] = (R[\alpha] \cap R[1/\alpha])[\alpha] \cap (R[\alpha] \cap R[1/\alpha])[1/\alpha] \subset R[\alpha] \cap R[1/\alpha] = R(\alpha)$.

LEMMA 1.9. *Let* $\alpha$ *be an anti-integral element in* $K$ *over* $R$. *Let* $\beta = u\alpha + a$ $(u, a \in R)$ *with* $u$ *an unit in* $R$. *Then* $\beta$ *is an anti-integral element.*

PROOF. It is easy to see that $u\alpha$ is anti-integral over $R$. So we may assume that $u = 1$ and $\beta = \alpha + a\ (a \in R)$. Let

$$0 \longrightarrow I \longrightarrow R[X] \xrightarrow{\phi} R[\beta] \longrightarrow 0$$

be an exact sequence, where $R[X]$ is a polynomial ring and $\phi(X) = \beta$. By [3], we have only to show that $I$ has a linear basis, that is, $I = \Sigma(c_i X - d_i)R[X]$ with $\beta = d_i/c_i\ (c_i, d_i \in R)$. Take $f(X) \in I \subset R[X]$. Putting $Y = X - a$, let $g(Y) = f(X)$. Since $f(\beta) = g(\alpha) = 0$ and $\alpha$ is anti-integral over $R$, we have $g(Y) = \Sigma(a_i Y - b_i)h_i(Y) = \Sigma(a_i(X - a) - b_i)h_i(X - a) = \Sigma(a_i X - (b_i + a_i a))h_i(X - a)$ and $(b_i + a_i a)/a_i = b_i/a_i + a = \alpha + a = \beta$. Thus $I$ has a linear basis, which implies that $\beta$ is anti-integral over $R$. Q.E.D.

PROPOSITION 1.10. *For a non-zero* $\alpha \in K$, *if* $R(\alpha) \supsetneqq R$ *there exists* $r \in R$ *such that* $r\alpha \in R(\alpha) \setminus R$.

PROOF. Take $\beta \in R(\alpha) \setminus R$. Then

$$(*) \qquad \beta = a_n \alpha^n + \cdots + a_0 = b_m (1/\alpha)^m + \cdots + b_0$$

with $a_i$, $b_j \in R$. Changing $\beta$ and the expression $(*)$, we may assume that $n$ is the smallest among such $n$'s. If $n = 1$, then we are done. Suppose $n > 1$. We can assume that $a_0 = 0$. Thus

$$\beta/\alpha = a_n \alpha^{n-1} + \cdots + a_1 = b_m (1/\alpha)^{m+1} + \cdots + b_0 (1/\alpha) \in R(\alpha).$$

By the choice of $n$, we have $\beta/\alpha \in R$, and hence $\beta = r\alpha$ with $r \in R$. Q.E.D.

COROLLARY 1.11. *For a non-zero* $\alpha \in K$, *let* $I_\alpha = q_1 \cap \cdots \cap q_t$ *be a primary decomposition. Then if* $\sqrt{q_i} \notin Ass_R(\bar{R}/R)$ *for each i, then* $\alpha$ *is anti-integral over* $R$.

PROOF. Suppose that $R(\alpha) \supsetneqq R$. Then there exists $r \in R$ such that $r\alpha \in R(\alpha) \setminus R$ by Proposition 1.10. Since $I_{r\alpha} = I_\alpha :_R r$, we have $I_{r\alpha} = (q_1 : r) \cap \cdots \cap (q_t : r)$. Since $r\alpha$ is integral over $R$ (Lemma 1.2), there exists $i$ such that $\sqrt{q_i} \in \mathrm{Ass}_R(\bar{R}/R)$. Q.E.D.

REMARK 1.12. Let $\alpha \in K$ and let $S$ be a multiplicatively closed subset of $R$. Then $(S^{-1}R)(\alpha) = S^{-1}(R(\alpha))$. If an over-ring $A$ is an anti-integral extension of

$R$, $S^{-1}A$ is an anti-integral extension of $S^{-1}R$ and a polynomial extension $A[X]$ is an anti-integral extension of $R[X]$.

## 2. Anti-integral extensions and flat simple extensions.

DEFINITION 2.1. Let $A = R[\alpha]$, $\alpha$ a non-zero element in $K$. For $p \in \operatorname{Spec} R$, we denote $A_p = R_p[\alpha]$. We say that an extension $A/R$ is a blowing-up at $p$ if the following two conditions are satisfied:
(i) $pA_p \cap R_p = pR_p$,
(ii) $A_p/pA_p$ is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

LEMMA 2.2. *Let $(R, m)$ be a local domain and $A = R[\alpha]$, $\alpha \in K$ $(\alpha \neq 0)$. Assume that $mA \cap R = m$. In this case we have a canonical injection $R/m \longrightarrow A/mA = (R/m)[\bar{\alpha}]$. Then*
*(1) $A/R$ is a blowing-up at $m$ if and only if $\bar{\alpha}$ is transcendental over $R/m$ (where $\bar{\alpha}$ denotes the residue class of $\alpha$ in $A/mA$).*
*(2) Assume that $R$ contains an infinite field $k$. If $A/R$ is not a blowing-up at $m$, then for some $\mu \in k$, $1/(\alpha - \mu)$ is integral over $R$ and $R \xrightarrow{i} R(\alpha - \mu) = R[1/(\alpha - \mu)] \xrightarrow{j} R[\alpha] = R[\alpha - \mu, 1/(\alpha - \mu)]$, where $i$ is integral and $j$ is a localization (flat).*

PROOF. (1) is obvious. (2): By the preceding (1), $\bar{\alpha}$ is algebraic over $R/m$. So we have

$$(**) \qquad \bar{\alpha}^k(\bar{\alpha}^m + \lambda_1' \bar{\alpha}^{m-1} + \cdots + \lambda_m') = 0 \qquad (\lambda_i' \in R/m,\ \lambda'_m \neq 0)$$

If $k > 0$, we can find $\mu \neq 0$ in the infinite field $k$ such that $\mu^m + \lambda_1' \mu^{m-1} + \cdots + \lambda_m' \neq 0$. So putting $\beta = \alpha - \mu$, from $(**)$ we have an algebraic relation:

$$\bar{\beta}^n + \lambda_1 \bar{\beta}^{n-1} + \cdots + \lambda_n = 0 \qquad (\lambda_n \neq 0,\ \lambda_i \in R/m).$$

Let $a_i$ denote the representative of $\lambda_i$ in $R$. Then we have

$$\beta^n + a_1 \beta^{n-1} + \cdots + a_n \in mR[\beta].$$

Since $\bar{a}_n = \lambda_n \neq 0$, we have $a_n \notin m$. By the expression of an element in $mR[\beta]$, we get

$$b_m \beta^m + \cdots + b_0 = 0$$

with $b_i \in R$ and $\bar{b}_0 = \bar{a}_n = \lambda_n \neq 0$. Hence $b_0$ is a unit in $R$. Multiplying $1/b_0$, we have

$$c_m\beta^m + \cdots + c_1\beta + 1 = 0$$

which implies that $1/\beta$ is integral over $R$. By Lemma 1.3, $R(\beta) = R(1/\beta) = R[1/\beta]$ and $R \xrightarrow{i} R(\beta) = R[1/\beta] \xrightarrow{j} R[\beta, 1/\beta]$, where $i$ is integral and $j$ is a localization (hence flat). Q.E.D.

DEFINITION 2.3. Let $R \longrightarrow A$ be a ring extension and $p \in \operatorname{Spec} R$. We say that $p$ is contracted from $A$ if $pA \cap R = p$.

It is easy to see that $p \in \operatorname{Spec} R$ is contracted from $A$ if and only if $P \cap R = p$ for some $P \in \operatorname{Spec} A$ if and only if $pA_p \cap R_p = pR_p$. If $A = R[\alpha]$ is a blowing-up at $p \in \operatorname{Spec} R$, then $p$ is contracted from $A$.

LEMMA 2.4. *Let $(R, m)$ be a local domain and $A = R[\alpha]$ with a non-zero $\alpha \in K$. Then*
(1) *$m$ is not contracted from $A$ if and only if $mA = A$.*
(2) *If $m$ is not contracted from $A$, then $1/\alpha$ is integral over $R$ and $R \xrightarrow{i} R(\alpha) = R[1/\alpha] \xrightarrow{j} R[\alpha] = R[\alpha, 1/\alpha]$, where $i$ is integral and $j$ is flat.*

PROOF. (1) is obvious. (2): Suppose that $mA = A$. Then

$$1 = a_0 + a_1\alpha + \cdots + a_n\alpha^n \qquad (a_i \in m).$$

Since $1 - a_0$ is a unit in $R$, $1/\alpha$ is integral over $R$. Hence noting that $1/\alpha \in R(\alpha) \subset R[\alpha]$, we have

$$R \xrightarrow{i} R(\alpha) = R[1/\alpha] \xrightarrow{j} R[\alpha, 1/\alpha] = R[\alpha],$$

where $i$ is integral and $j$ is a localization. Q.E.D.

PROPOSITION 2.5. *Let $(R, m)$ be a local domain and $\alpha$ a non-zero element in $K$ which is anti-integral over $R$. Assume that $R[\alpha] \supsetneqq R$. Then $mR[\alpha] = R[\alpha]$ if and only if $R = R(\alpha) = R[1/\alpha] \xrightarrow{j} R[\alpha]$, where $j$ is flat.*

PROOF. ($\Rightarrow$): By Lemma 2.4, $1/\alpha$ is integral over $R$ and $R(\alpha) = R(1/\alpha) = R[1/\alpha] \xrightarrow{j} R[\alpha]$, where $j$ is flat. Since $\alpha$ is anti-integral, $R(\alpha) = R$. ($\Leftarrow$): By assumption, $\alpha \notin R$. Since $1/\alpha \in R$, we have $1/\alpha \in m$, which implies that $mR[\alpha] = R[\alpha]$. Q.E.D.

The ring extension $R[\alpha]$ of $R$ is characterized by an ideal $I_\alpha$ as is seen in the following Proposition.

PROPOSITION 2.6. *Let $\alpha \in K$ be anti-integral over $R$ and $p \in \operatorname{Spec} R$. Assume that $R$ contains an infinite field $k$. Then*
(1) *$I_\alpha \not\subset p$ if and only if $R_p = R_p[\alpha]$.*
(2) *$I_\alpha \subset p$ but $\alpha I_\alpha \not\subset p$ if and only if $R_p \neq R_p[\alpha]$ and $p$ is not contracted from $R[\alpha]$.*
(3) *$I_\alpha + \alpha I_\alpha \subset p$ if and only if $R[\alpha]/R$ is blowing-up at $p$.*

PROOF. (1): $I_\alpha \not\subset p \Leftrightarrow \alpha \in R_p \Leftrightarrow R_p[\alpha] = R_p$.
(2) ($\Rightarrow$): Since $\alpha I_\alpha \not\subset p$, there exists $b \in \alpha I_\alpha$ but $b \notin p$. Hence $\alpha = b/a$ for some $a \in I_\alpha \subset p$. So $a\alpha = b \in pR[\alpha]$ and hence $pR[\alpha] \cap R \supsetneqq p$. ($\Leftarrow$): By Lemma 2.4, $R_p \xrightarrow{i} R_p[1/\alpha] \xrightarrow{j} R_p[\alpha]$, where $i$ is integral and $j$ is flat, and $1/\alpha$ is integral over $R_p$. Since $\alpha$ is anti-integral over $R_p$, we have $R_p = R_p(\alpha) = R_p(1/\alpha) = R_p[1/\alpha]$. Hence $1/\alpha \in R_p$, which implies that $\alpha I_\alpha = I_{1/\alpha} \not\subset p$. Since $pR_p[\alpha] = R_p[\alpha]$, we have $R_p[\alpha] \supsetneqq R_p$ and hence $I_\alpha \subset p$.
(3) ($\Rightarrow$): Since $R_p[\alpha]/pR_p[\alpha] = (R_p/pR_p)[\bar{\alpha}]$, we must show that $\bar{\alpha}$ is transcendental over $R_p/pR_p$. Suppose the contrary. We have by Lemma 2.2, that $1/\beta$ is integral over $R_p$ with $\beta = \alpha - \mu$ for some $\mu \in k$ and by Lemma 1.9 $\beta$ is anti-integral over $R$. So $1/\beta \in R_p$. Hence $\beta I_\beta = I_{1/\beta} \not\subset p$. But it is easy to see that $I_\alpha + \alpha I_\alpha = I_\alpha(R + \alpha R) = I_\beta(R + \beta R) = I_\beta + \beta I_\beta$, which yields that $I_\beta + \beta I_\beta \subset p$, contradiction. ($\Leftarrow$): Suppose that $I_\alpha + \alpha I_\alpha \not\subset p$. If $I_\alpha \not\subset p$, then $R_p = R_p[\alpha]$ and $R[\alpha]$ is not a blowing-up at $p$ by (1). If $I_\alpha \subset p$ and $\alpha I_\alpha \not\subset p$, then $pR_p[\alpha] \cap R_p \neq pR_p$ by (2). Thus $R[\alpha]$ is not a blowing-up at $p$ by Definition 2.1. Therefore in any case $R[\alpha]$ is not a blowing-up at $p$. Q.E.D.

In [7], the following result is shown:

Let $A$ be an integral domain (not necessarily Noetherian), $L$ its quotient field and $\alpha = b/a \in L (a, b \in A, ab \neq 0)$. Then if $A[\alpha]$ is flat over $A$ and $A$ is integrally closed in $L$, an ideal $(a, b)A$ is invertible.

The analogous result is the following:

PROPOSITION 2.7. *Let $\alpha \in K$ be a non-zero element. Then*
(1) *If $I_\alpha + \alpha I_\alpha = R$ then $R[\alpha]$ is flat over $R$.*
(2) *Assume that $\alpha = b/a$ $(a, b \in R)$ is anti-integral over $R$. Then the following statements are equivalent:*
(2.i) *$R[\alpha]$ is flat over $R$.*
(2.ii) *$I_\alpha + \alpha I_\alpha = R$.*
(2.iii) *$(a, b)R$ is an invertible ideal of $R$.*

PROOF. Since $I_\alpha(R + \alpha R) = R$, a fractional ideal $R + \alpha R$ is invertible. Since

$R+\alpha R \simeq a(R+\alpha R)=(a, b)R$, $(a, b)R$ is invertible. So $R[\alpha]$ is flat (cf. [7]).
(2): (2.iii) $\Rightarrow$ (2.i) follows from [7]. (2.ii) $\Rightarrow$ (2.iii) is shown above. (2.i) $\Rightarrow$ (2.ii): Note that $p \supset I_\alpha + \alpha I_\alpha$ if and only if $R[\alpha]/R$ is a blowing-up at $p$ by Proposition 2.6. Since $R[\alpha]$ is flat over $R$, $R[\alpha]/R$ is not a blowing-up at any $p \in \operatorname{Spec} R$. Thus $I_\alpha + \alpha I_\alpha = R$. Q.E.D.

PROPOSITION 2.8. *Let $\alpha \in K$ be anti-integral over $R$. Then $R[\alpha]$ is flat over $R$ if and only if $R[1/\alpha]$ is flat over $R$.*

PROOF. By Proposition 2.7, $R[\alpha]$ is flat over $R \Leftrightarrow I_\alpha + I_{1/\alpha} = I_\alpha + \alpha I_\alpha = R$ $\Leftrightarrow I_\alpha + I_{1/\alpha} = (1/\alpha)I_{1/\alpha} + I_{1/\alpha} = R$ (Note that $\alpha I_\alpha = I_{1/\alpha}$) $\Leftrightarrow R[1/\alpha]$ is flat over $R$. Q.E.D.

LEMMA 2.9. *Let $\alpha \in K$ $(\alpha \neq 0)$ and $P \in \operatorname{Spec} R(\alpha)$. Put $p = P \cap R$. Then either $pR[\alpha] \neq R[\alpha]$ or $pR[1/\alpha] \neq R[1/\alpha]$.*

PROOF. Suppose $pR[\alpha] = R[\alpha]$. Then since $pR_p[\alpha] = R_p[\alpha]$, we have $R_p(\alpha) = R_p[1/\alpha]$ by Lemma 2.4. Hence $R_p(\alpha) \supsetneqq PR_p(\alpha) \supset pR_p(\alpha) = pR_p[1/\alpha]$ and consequently we have $pR_p[1/\alpha] \neq R_p[1/\alpha]$. Thus $pR[1/\alpha] \neq R[1/\alpha]$.
Q.E.D.

The next result is well-known (cf. [7]).

LEMMA 2.10. *Let $A$ be an over-ring of $R$. Then $A$ is flat over $R$ if and only if $A_P = R_{P \cap R}$ for each $P \in \operatorname{Spec} A$.*

PROPOSITION 2.11. *Let $\alpha$ be a non-zero element in $K$. Assume that both $R[\alpha]$ and $R[1/\alpha]$ are flat over $R$. Then $R(\alpha) = R$, i.e., $\alpha$ is anti-integral over $R$.*

PROOF. Take $P \in \operatorname{Spec} R(\alpha)$ and put $p = P \cap R$. Then by Lemma 2.9, either $pR[\alpha] \neq R[\alpha]$ or $pR[1/\alpha] \neq R[1/\alpha]$. We may assume $pR[\alpha] \neq R[\alpha]$ (by symmetry). Then there exists $Q \in \operatorname{Spec} R[\alpha]$ such that $Q \cap R = p$. Since $R_p[\alpha]$ is flat over $R_p$, $R_p = (R_p[\alpha])_Q \supset R_p[\alpha] \supset R_p$. Hence $R_p[\alpha] = R_p$. Thus $R_p(\alpha) = R_p[\alpha] \cap R_p[1/\alpha] = R_p$. Since $R(\alpha)$ is integral over $R$ (Lemma 1.2), we have $R(\alpha)_P = R_p$, which implies that $R(\alpha)$ is flat over $R$. So since $R(\alpha)$ is integral over $R$, $R(\alpha) = R$ by [7]. Q.E.D.

COROLLARY 2.12. *Let $\alpha = b/a \in K$ $(a, b \in R, ab \neq 0)$. Then if both $R[\alpha]$ and $R[1/\alpha]$ are flat over $R$, $(a, b)R$ is an invertible ideal of $R$.*

PROOF. By Lemma 2.11, $\alpha$ is anti-integral over $R$. The conclusion follows from Proposition 2.7. Q.E.D.

COROLLARY 2.13. *Let $\alpha \in K$ be a non-zero element. Consider the following conditions*:
(1) $R[\alpha]$ *is an anti-integral extension of* $R$.
(2) $R[\alpha]$ *is flat over* $R$.
(3) $R[1/\alpha]$ *is flat over* $R$.
*Then two of these conditions imply the other.*

PROOF. If (1) holds, (2) $\Leftrightarrow$ (3) follows from Proposition 2.8. (2) + (3) implies (1) by Proposition 2.11. Q.E.D.

We recall that: Let $A$ be an over-ring of $R$ which is of finite type over $R$. Assume that $R$ is integrally closed in $K$ and that $A$ is quasi-finite over $R$ (cf. [6]). Then the contraction map $\phi$: Spec $A \longrightarrow$ Spec $R$ is an open immersion, and if furthermore $\phi$ is surjective, then $A = R$.

This is known as the Zariski-Main Theorem (ZMT) (cf. [6]). We extend this in a certain sense (See Theorem 2.16). We start with a simple case.

PROPOSITION 2.14. *Let $\alpha$ be a non-zero element in $K$ which is anti-integral over $R$. Assume that $R$ contains an infinite field $k$ and that $R$ satisfies the altitude formula. Then if the contraction map $\phi$ : Spec $R[\alpha] \longrightarrow$ Spec $R$ is such that Im $\phi$ $\supset Dp_1(R)$, then $R[\alpha] = R$.*

PROOF. Take $p \in \operatorname{Spec} R$. If $I_\alpha \not\subset p$ then $R[\alpha] \subset R_p$, and hence $R_p[\alpha] = R_p$. Suppose $I_\alpha \subset p$ and that $p$ is a prime divisor of $I_\alpha$. Then $p \in \mathrm{Dp}_1(R)$. So since Im $\phi \supset \mathrm{Dp}_1(R)$, we have $\alpha I_\alpha \subset p$ by Proposition 2.6. Thus $\sqrt{I_\alpha} \supset \sqrt{\alpha I_\alpha}$. Note that $I_\alpha = I_{\alpha-\mu}$ for any $\mu \in k$ and $R[\alpha] = R[\alpha - \mu]$ and that $\alpha - \mu$ is anti-integral over $R$ by Lemma 1.9. We have $\sqrt{I_{\alpha-\mu}} \supset \sqrt{(\alpha-\mu) I_{\alpha-\mu}}$ for any $\mu \in k$. Now let $q$ be in $\mathrm{Ht}_1(R)$ and suppose $q \supset I_\alpha$. Then $R[\alpha]/R$ is not a blowing-up at $q$. Indeed, suppose that $R_q[\alpha]/qR_q[\alpha] \simeq (R_q/qR_q)[T]$, $T$ an indeterminate. Then $qR_q[\alpha]$ is a prime ideal and $\mathrm{ht}\, qR_q[\alpha] = \mathrm{ht}\, qR_q + \mathrm{tr.deg}_R R[\alpha] - \mathrm{tr.deg}_{k(q)} R_q[\alpha]/qR_q[\alpha] = 1 + 0 - 1 = 0$ $(k(q) = R_q/qR_q)$ by the altitude formula, which implies that $qR_q[\alpha] = 0$, contradiction. Hence $1/\beta$ is integral over $R$ for $\beta = \alpha - \mu$ (some $\mu \in k$) by Lemma 2.2. Since $\beta$ is anti-integral over $R$ by Lemma 1.9, $1/\beta \in R_q$. Thus $\beta I_\beta = I_{1/\beta} \not\subset q$, which contradicts with $q \supset \sqrt{I_\beta} \supset \sqrt{\beta I_\beta}$. Therefore $\mathrm{ht}\, I_\alpha > 1$. But since $R$ satisfies the altitude formula, the element $\alpha$ is integral over $R$. Indeed, putting $p = P \cap R$ for arbitrary $P \in \mathrm{Ht}_1(\bar{R})$, we have $\mathrm{ht}\, p = 1$ by the altitude formula. Thus $I_\alpha \not\subset p$ and hence $\alpha \in R_p \subset \bar{R}_p$, which yields that $\alpha \in \bar{R}$. Therefore $R(\alpha) = R[\alpha]$ by Lemma 1.3. So $R[\alpha] = R(\alpha) = R$.
Q.E.D.

As a corollary of the proof of Proposition 2.14, we have the following:

COROLLARY 2.15. *Let* $\alpha$ *be an anti-integral element in* $K$. *Assume that* $R$ *contains an infinite field* $k$ *and that* $R[\alpha]$ *is quasi-finite over* $R$. *Then if the contraction map* $\phi : \mathrm{Spec}\, R[\alpha] \longrightarrow \mathrm{Spec}\, R$ *is such that* $\mathrm{Im}\, \phi \supset Dp_1(R)$, *then* $R[\alpha] = R$.

PROOF. By the same way as in the proof of Proposition 2.14, we can show that $\sqrt{I_\alpha} \supset \sqrt{\alpha I_\alpha}$. Since $R[\alpha]$ is quasi-finite over $R$, $R[\alpha]$ is not a blowing-up at any $p \in \mathrm{Spec}\, R$ with $pR[\alpha] \neq R[\alpha]$. Hence by Proposition 2.6(3). $I_\alpha + \alpha I_\alpha \not\subset p$ for any $p \in \mathrm{Spec}\, R$ with $pR[\alpha] \neq R[\alpha]$. So $\sqrt{I_\alpha} \supset \sqrt{\alpha I_\alpha}$ implies that $I_\alpha \not\subset p$ for any $p \in \mathrm{Spec}\, R$ with $pR[\alpha] \neq R[\alpha]$. Thus $\alpha \in R_p$ for any $p \in \mathrm{Spec}\, R$ with $pR[\alpha] \neq R[\alpha]$. But since $\mathrm{Im}\, \phi \supset \mathrm{Dp}_1(R)$, we have $pR[\alpha] \neq R[\alpha]$ for any $p \in \mathrm{Dp}_1(R)$ and hence $R = \cap R_p \supset \cap R_p[\alpha] \supset R[\alpha]$ ($p \in \mathrm{Dp}_1(R)$), which implies that $R = R[\alpha]$. Q.E.D.

THEOREM 2.16. *Assume that* $R$ *contains an infinite field* $k$ *and that* $R$ *satisfies the altitude formula. Let* $A = R[\alpha_1, \ldots, \alpha_n]$ *with* $\alpha_i \in K$ *anti-integral over* $R$. *Then if the contraction map* $\phi : \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ *satisfies* $\mathrm{Im}\, \phi \supset Dp_1(R)$, *then* $A = R$.

PROOF. From the commutative diagram:

$$\begin{tikzcd} \mathrm{Spec}\, A \arrow[rr] \arrow[dr, "\phi"'] & & \mathrm{Spec}\, R[\alpha_i] \arrow[dl, "\phi_i"] \\ & \mathrm{Spec}\, R & \end{tikzcd}$$

we have $\mathrm{Im}\, \phi_i \supset \mathrm{Im}\, \phi \supset \mathrm{Dp}_1(R)$. Thus $\alpha_i \in R$ for each $i$ by Proposition 2.14. So $A = R$. Q.E.D.

COROLLARY 2.17. *Under the same assumption as in Theorem* 2.16, *the following statements are equivalent:*

(1) $\phi$ *is a surjection.*

(2) $A$ *is integral over* $R$.

(3) $A = R$.

COROLLARY 2.18. *Let* $A = R[\alpha_1, \ldots, \alpha_n]$ *with non-zero* $\alpha_i$ *anti-integral over* $R$. *Assume that* $R$ *contains an infinite field* $k$ *and that* $A$ *is quasi-finite over* $R$. *Then if the contraction map* $\phi : \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ *satisfies* $\mathrm{Im}\, \phi \supset Dp_1(R)$, *then* $A = R$.

REMARK 2.19. Let $A$ and $R$ be the same as in Thoerem 2.16 or Corollary 2.18. If $A$ is integral over $R$, $\phi : \mathrm{Spec}\, A \longrightarrow \mathrm{Spec}\, R$ is obviously surjective. But the surjectivity of $\phi$ does not necessarily ensure the integrality of $A$ over $R$. For example: $A = k[T, 1/(T-1)] \supset k[T] \supset R = k[T(T-1), T^2(T-1)]$, where $k$ is a field. Theorem 2.16 asserts that if $A$ is anti-integral over $R$, the surjectivity of $\phi$ ensures the integrality of $A$ over $R$.

**3. The ring $R(I_\alpha)$ and the anti-integrality of $\alpha$.**

We introduce the following over-ring of $R$.

DEFINITION 3.1. Let $J$ be a fractional ideal of $R$. Define

$$R(J) = \{\alpha \in K \mid \alpha J \subset J\}.$$

It is easy to see that $R(J)$ is an over-ring of $R$ which is integral over $R$ and $R(J_p) = R(J)_p$ for any $p \in \mathrm{Spec}\, R$ (cf. [5]).

PROPOSITION 3.2. *Let $J$ be an integral ideal of $R$. Then the following statements are equivalent:*

(1) $R(J^{-1}) = R$ *(where $J^{-1} = (R :_K J) = \mathrm{Hom}_R(J, R)$).*

(2) $J_p$ *is a principal ideal of $R_p$ for each $p \in \mathrm{Dp}_1(R)$.*

PROOF. (2) $\Rightarrow$ (1): Since $R = \cap R_p$ $(p \in \mathrm{Dp}_1(R))$, we have only to show that $R(J^{-1}) \subset R_p$ for any $p \in \mathrm{Dp}_1(R)$. As $J_p = \alpha R_p$ for some $\alpha \in R$, $J_p^{-1} = \alpha^{-1} R_p$. So we have $R_p \subset R(J_p^{-1}) = R(\alpha^{-1} R_p) \subset R_p$. The containment $R(J^{-1}) \subset R(J_p^{-1})$ is obvious. Thus $R(J^{-1}) \subset R(J_p^{-1}) \subset R_p$, as wanted. (1) $\Rightarrow$ (2): We may assume that $(R, m)$ is a local domain and that $m \in \mathrm{Dp}_1(R)$. If $R$ is a discrete valuation ring, $R$ is a principal ideal domain. Hence $J$ is a principal ideal of $R$. Suppose that $R$ is not a discrete valuation ring. Then $m :_K m \neq R$. If there exists $\phi \in J^{-1} = \mathrm{Hom}_R(J, R)$ with $\phi(J) = R$, an exact sequence: $J \xrightarrow{\phi} R \longrightarrow 0$ splits. Since rank $J = 1$, we have $J \simeq R$. So we suppose that $\phi(J) \neq R$ for any $\phi \in J^{-1}$. In this case, $\phi(J) \subset m$. For any $\alpha \in m :_K m$, $\alpha\phi(J) \subset \alpha m \subset m \subset R$, and hence $\alpha \in R(J^{-1})$. Thus $m :_K m \subset R(J^{-1})$. But since $R(J^{-1}) \supset m :_K m \supsetneqq R$, we have $R = R(J^{-1}) \supsetneqq R$, contradiction. Q.E.D.

PROPOSITION 3.3. *Let $J$ be divisorial integral ideal of $R$. Then if $R(J^{-1}) = R$, there exists a non-zero $\alpha \in K$ such that $J = I_\alpha$.*

PROOF. We first show the following Claim:

CLAIM. If $p \in \mathrm{Dp}_1(R)$ contains $J$, then $p$ is a prime divisor of $J$.

(Proof) Since $R(J^{-1}) = R$, $J_p = aR_p$ for some $a \in J$ by Proposition 3.2. As $pR_p$ is a prime divisor of a principal ideal $aR_p = J_p$ (cf. [8]), $p$ is a prime divisor of $J$.

Let $J = q_1 \cap \cdots \cap q_n$ ($\sqrt{q_i} = p_i$) be a primary decomposition. Then $p_i \in \mathrm{Dp}_1(R)$ because $J$ is divisorial. Put $S = R \setminus \cup p_i$. Then $A := S^{-1}R$ is semi-local and each maximal ideal of $A$ is of depth one. Since $JA$ is invertible by Claim above, and since $A$ is semi-local, $JA$ is a principal ideal. So there exists $a \in J$ such that $JA = aA$. Hence we have a primary decomposition:

$$aR = q_1 \cap \cdots \cap q_n \cap Q_1 \cap \cdots \cap Q_m \quad (\sqrt{Q_i} = P_i, P_i \neq p_1, \dots, p_n)$$

Suppose $P_i \supset J$ for some $i$. Then $P_i \in \mathrm{Dp}_1(R)$ implies that $P_i$ is a prime divisor of $J$ by Claim. But since the prime divisors of $J$ are $p_1, \dots, p_n$, this is absurd. Hence $J \not\subset P_i$ for all $i = 1, \dots, m$. Next we will show that $P_i \not\subset p_j$ for any $i, j$. Suppose that $P_i \subset p_j$. Then $JR_{p_j} = aR_{p_j}$, and $a \in P_i$ implies $JR_{p_j} \subset P_iR_{p_j}$. Thus $J \subset P_i$, contradiction. Hence we have

$$Q_1 \cap \cdots \cap Q_m \not\subset p_1, \dots, p_n.$$

So there exists $b \in Q_1 \cap \cdots \cap Q_m$, $b \notin p_1, \dots, p_n$. We have $(aR :_R bR) = q_1 \cap \cdots \cap q_n = J$. Putting $\alpha = b/a \in K$, we have $I_\alpha = (aR :_R bR) = J$. Q.E.D.

PROPOSITION 3.4. *Let $\alpha$ be a non-zero element in $K$. If $R(I_\alpha) = R$, then $R(\alpha) = R$, i.e., $\alpha$ is anti-integral over $R$.*

PROOF. CLAIM 1: For each $p \in \mathrm{Dp}_1(R)$, $(I_\alpha)_p = R_p$ or $(I_{1/\alpha})_p = R_p$. (Proof) Let $p \in \mathrm{Dp}_1(R)$. If $I_\alpha \not\subset p$ then $(I_\alpha)_p = R_p$. Suppose that $I_\alpha \subset p$ and $I_{1/\alpha} \subset p$. Then $(I_\alpha)_p = aR_p$ for some $a \in I_\alpha$ by Proposition 3.2. So $\alpha a R_p = \alpha(I_\alpha)_p = (\alpha I_\alpha)_p = (I_{1/\alpha})_p \subset pR_p$. Thus $\alpha a = r$ for some $r \in pR_p$ and $\alpha = r/a$. So $aR_p = (I_\alpha)_p = (aR_p :_{R_p} rR_p)$, which asserts that $a, r$ is an $R_p$-regular sequence. But depth $R_p = 1$, contradiction. Therefore $(I_\alpha)_p = R_p$ or $(I_{1/\alpha})_p = R_p$.

CLAIM 2: If $R(I_\alpha) = R$ then $I_\alpha + I_{1/\alpha} \not\subset p$ for each $p \in \mathrm{Dp}_1(R)$. (Proof) Suppose that $I_\alpha + I_{1/\alpha} \subset p$ with $p \in \mathrm{Dp}_1(R)$. Then both $(I_\alpha)_p$ and $(I_{1/\alpha})_p$ are principal ideals in $R_p$ by Proposition 3.2. But by Claim 1, $(I_\alpha)_p = R_p$ or $(I_{1/\alpha})_p = R_p$. This contradicts with $(I_\alpha)_p + (I_{1/\alpha})_p \subset pR_p$.

Thus for $p \in \mathrm{Dp}_1(R)$, $I_\alpha + I_{1/\alpha} \not\subset p$ by Claim 2. Hence $\alpha \in R_p$ or $1/\alpha \in R_p$. Therefore $R(\alpha) = R[\alpha] \cap R[1/\alpha] \subset R_p$ for all $p \in \mathrm{Dp}_1(R)$. So $R(\alpha) = R$ by Lemma 1.5. Q.E.D.

COROLLARY 3.5. *Let $J$ be an invertible ideal contained in $R$. Then there exists an anti-integral element $\alpha$ of $K$ such that $J = I_\alpha$.*

PROOF. Since $R(J^{-1}) = R$ by Proposition 3.2, $J = I_\alpha$ for some non-zero $\alpha$ in

$K$ by Proposition 3.2. Hence $R(\alpha)=R$ by Proposition 3.4. Q.E.D.

REMARK 3.6. The converse of Proposition 3.4 does not necessarily hold as is seen in the following example:

Let $(R, m)$ be a Noetherian local domain such that $\mathrm{Ass}_R(\bar{R}/R)=\{m\}$, ht $m>1$. Take $\alpha \in K$ such that $I_\alpha = q_1 \cap \cdots \cap q_t$, where $q_i$ is a primary ideal with ht $q_i = 1$ for each $i$. Then $\sqrt{q_i} \notin \mathrm{Ass}_R(\bar{R}/R)$ implies that $R(\alpha)=R$ by Corollary 1.11. But since depth $R = 1$, $I_\alpha$ can not be a principal ideal, and hence $R(I_\alpha) \neq R$ (cf. Proposition 3.2).

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