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ON THE HOMOTOPY OF LINKS

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§0. Introduction.

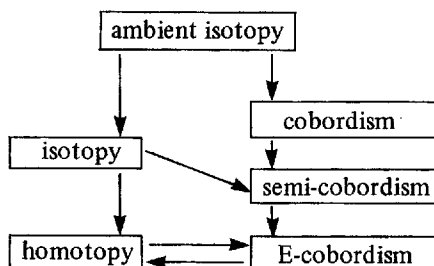
Let ℓ be a tame oriented link in a 3-space R^3 . ℓ is said to be *E-cobordant* to a link ℓ' (or ℓ and ℓ' are *E-cobordant*), denoted by $\ell \sim_E \ell'$, if there are mutually disjoint proper annuli $\mathcal{A}$ in $R^3[a, b] = \{(x, y, z, t) \in R^4 \mid a \leq t \leq b\}$ satisfying the followings:

- (i) $\mathcal{A} \cap R^3[a] = \ell$, $\mathcal{A} \cap R^3[b] = -\ell'$ and each of $\mathcal{A}$ contains a component of ℓ and one of $-\ell'$, where $-\ell'$ means the reflective inverse of ℓ' .
 - (ii) each annulus A of $\mathcal{A}$ is non-singular except for finite points $P_1, \dots, P_n$ in the interior of A such that for $B_i^4 = N(P_i; R^4)$, $(\partial B_i^4, \partial B_i^4 \cap A)$ is a Hopf link, [13], where $N(x; X)$ means the regular neighborhood of x in X .
- $\mathcal{A}$ is called *E-annuli* (between ℓ and ℓ').

Especially if *E-annuli* $\mathcal{A}$ are level preserving, i.e., $\mathcal{A}$ have neither minimal nor maximal points, ℓ is said to be *homotopic* to ℓ' (or ℓ and ℓ' are *homotopic*) and moreover each annulus of $\mathcal{A}$ is non-singular, ℓ is said to be *isotopic* to ℓ' (or ℓ and ℓ' are *isotopic*) and each annulus is non-singular locally flat, ℓ is said to be *ambient isotopic* to ℓ' (or ℓ and ℓ' are *ambient isotopic*) and denoted by $\ell \approx \ell'$.

If *E-annuli* $\mathcal{A}$ are non-singular, ℓ is said to be *semi-cobordant* to ℓ' (or ℓ and ℓ' are *semi-cobordant*) and denoted by $\ell \sim_{\text{semi}} \ell'$. Especially if $\mathcal{A}$ are locally flat and non-singular, ℓ is said to be *cobordant* to ℓ' (or ℓ and ℓ' are *cobordant*) and denoted by $\ell \sim \ell'$, [2], [13].

We know that cobordism implies homotopy by [4], [5] and *E-cobordism* is equivalent to homotopy by [13]. Therefore the relations among these concepts are the following.



Let ℓ and ℓ' be E -cobordant (or homotopic) links and $\mathfrak{A}$ the annuli between ℓ and ℓ' . If there is a point P in the interior of $\mathfrak{A}$ such that P is not locally flat, we modify $N(P; \mathfrak{A})$ in $N(P; R^4)$ such that P is locally flat point by adding to the singularity of $\mathfrak{A}$, see the proof of Theorem 1.4 in [13]. Hence we assume that if ℓ and ℓ' are E -cobordant (or homotopic), the E -annuli between ℓ and ℓ' are locally flat from now on.

In this paper, we consider the homotopy of links.

In §1, we study the homotopy of compound links and prove that the homotopy type of compound links of a faithful pair is same in Theorem 1.3, Corollary 1.5. In §2, we define a fusion between two ordered links which are splitted to each other and consider the homotopy of links obtained by the fusion. In §3, we consider some extensions of §1 and prove Theorem 3.1, 3.4 and 3.5.

Throughout this paper, we consider in the piecewise linear category.

§1. Homotopy of compound links.

Let V^* be a trivial solid torus and f a faithful homeomorphism of V^* onto a solid torus V . For a link ℓ^* in V^* , we denote $f(\ell^*)$ by ℓ and ℓ^* , ℓ are called the links of a *faithful pair*. And $o(\ell)$ means the minimum of number of $\ell \cap M$ for a meridian disk M in V .

By [11], [13] and [14], the followings are known.

LEMMA 1.1. ([11], [14]). *Let ℓ^*, ℓ and V^*, V be those of the above and c^*, c cores of V^*, V respectively. If $c \sim c^*$, then $\ell \sim \ell^*$.*

LEMMA 1.2. ([13]). *Homotopy is equivalent to E -cobordism.*

By Lemmas 1.1 and 1.2, we obtain the following.

THEOREM 1.3. *Let ℓ^* and ℓ be links of a faithful pair. Then ℓ is homotopic to ℓ^* .*

PROOF. If $o(\ell) = 0$, ℓ is ambient isotopic to ℓ^* in R^3 .

Hence we assume that $p = o(\ell) > 0$. Take a solid torus U with a core $-c$ which is splitted from V in R^3 . Let L be a non-twisted parallel link each of which is parallel to $-c$ with p -component in U such that $I(\ell, M_V) = -I(L, M_U)$, where M_V, M_U are the meridian disks of V, U respectively and $I(x, X)$ means the intersection number of x in X , see Fig. 1 (a). Since V and U are splitted, $\ell \cup L$ is splitted, denoted by $\ell \circ L$. We can assume that $\#(\ell \cap M_V) = \#(L \cap M_U) = p$ by choosing M_V, M_U suitably, where $\#(x)$ means the number of points. Take a 3-ball B in $R^3 - V - U$ such that $B \cap V, B \cap M_V$ consist of a disk, an arc and also

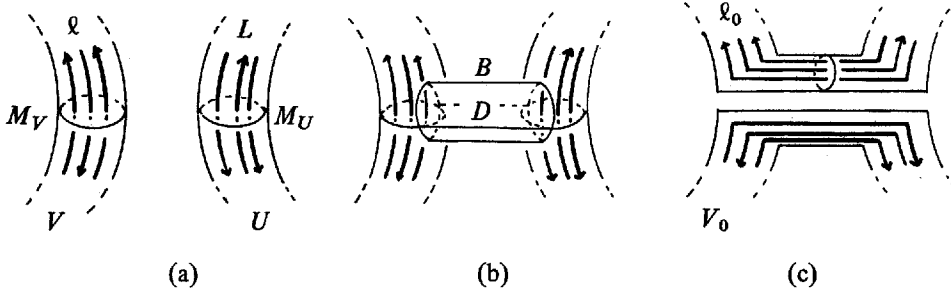


Fig. 1

$B \cap U$, $B \cap M_U$ a disk, an arc respectively. Let D be a proper disk in B such that $D \cap V = D \cap M_V$ and $D \cap U = D \cap M_U$, see Fig. 1 (b), and let $M = M_V \cup M_U \cup D$. Then M is a proper disk in $W = V \cup U \cup B$. Perform the fusion of $\ell \circ L$ (see the definition in §2) illustrated in Fig. 1 (c) in $V_0 = W - N(M : W)$ which is a solid torus and hence we obtain a link ℓ_0 in V_0 . As the core of V_0 is the product of c and $-c$ which is cobordant to zero, ℓ_0 is cobordant to ℓ_0^* for a faithful pair ℓ_0^*, ℓ_0 by Lemma 1.1. As L is non-twisted and parallel, L^* is a trivial link for a faithful pair L^*, L . Hence ℓ_0^* is ambient isotopic to ℓ^* . Moreover as L is non-twisted and parallel, L is a boundary link and so L is homotopic to zero (namely L is homotopic to a trivial link) by [1], [7], [8] and [15]. As L is splitted from ℓ , we see that $\ell \sim_E \ell^*$.

Therefore ℓ is homotopic to ℓ^* by Lemma 1.2.

COROLLARY 1.4. *For links ℓ^*, ℓ of a faithful pair, ℓ is homotopic to zero if and only if ℓ^* is homotopic to zero.*

Let f be a faithful homeomorphism of a solid torus V onto V' . For a link ℓ in V , we denote $f(\ell)$ by ℓ' . Then,

COROLLARY 1.5. *Let ℓ, ℓ' be those of the above. Then ℓ is homotopic to ℓ' .*

REMARK 1.6. There are links ℓ, ℓ^* such that ℓ is not isotopic to ℓ^* by [8] and that ℓ is not semi-cobordant to ℓ^* by [13].

§2. Homotopy of links obtained by a fusion.

Let ℓ be a link with m -component and $\mathfrak{B} = B_1 \cup \dots \cup B_n$ mutually disjoint n disks such that $B_i \cap \ell = \partial B_i \cap \ell$ which consists of two arcs with orientation coherently for $i = 1, \dots, n$. If $\ell' = \ell + \partial \mathfrak{B}$ is a link with $(m-n)$ -component (or

($m+n$)-component), we say that ℓ' is obtained by a *fusion* (or a *fission*) of ℓ and that $\mathfrak{B}$ are the *disks of fusion* (or *fission*), where $+$ means the homological addition.

Let $L = K_1 \cup \dots \cup K_n$, $L' = K'_1 \cup \dots \cup K'_n$ be ordered n -component links such that $L \cup L' \approx L \circ L'$ in R^3 and $\mathfrak{B} = B_1 \cup \dots \cup B_n$ the disks of fusion of $L \circ L'$ such that $B_i \cap K_i \neq \emptyset$, $B_i \cap K'_i \neq \emptyset$ for each $i = 1, \dots, n$. Then the link obtained by this fusion is denoted by $L \# L'$ or simply by $L \# L'$ and $L \# L'$ is

called a *link obtained by a fusion between L and L'* and $\mathfrak{B}$ are called *the disks of fusion between L and L'* . For example the links ℓ_1, ℓ_2 illustrated in Fig. 2 (b), (c) are obtained by fusions between a link L and its reflective inverse $-L$. For two

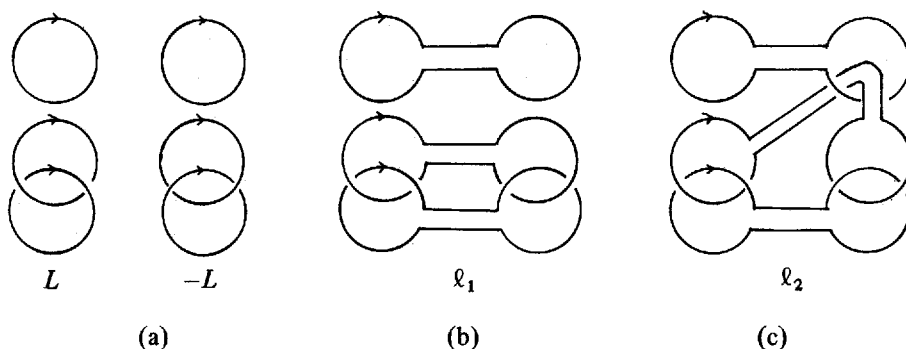


Fig. 2

knots K and K' , $K \# K'$ is unique under cobordism, [3], [12]. But unfortunately $L \# L'$ is not only unique under cobordism but also under homotopy which is pointed out by Y. Nakanishi. For example ℓ_1 is homotopic to zero and ℓ_2 is the Borromean rings which is not homotopic to zero by Theorem 5.1 in [9] and the Borromean rings is not cobordant to zero. (If L, L' are links with 2-component, $L \# L'$ is unique under homotopy by [7], [8]. But even if L, L' are 2-component, $L \# L'$ is not unique under cobordism.)

First of all, we prove the following.

THEOREM 2.1. *Let ℓ, ℓ' be links in R^3 . Then ℓ is homotopic to ℓ' if and only if there is a fusion between ℓ and $-\ell'$ such that $\ell \# (-\ell')$ is homotopic to zero.*

To prove Theorem 2.1, we prepare some lemmas.

Let $\ell = k_1 \cup \dots \cup k_n$ and $\ell' = k'_1 \cup \dots \cup k'_n$ be homotopic links and $\mathfrak{A} = A_1 \cup \dots \cup A_n$ level preserving E -annuli between ℓ and ℓ' . Recall that $\mathfrak{A}$ are locally flat. The singularity of $\mathfrak{A}$, A_i are denoted by $S(\mathfrak{A})$, $S(A_i)$ respectively.

By Lemma 4 in [10], we obtain the following.

LEMMA 2.2. *Let ℓ, ℓ' and $\mathfrak{A}, A_i$ be those of the above. Then ℓ is homotopic to ℓ' if and only if there is a non-singular locally flat surface $\mathfrak{F} = F_1 \cup \dots \cup F_n$ of genus zero in $R^3[0, 2]$ satisfying the followings:*

- (i) $\mathfrak{F}$ has neither minimal nor maximal points.
- (ii) $\partial \mathfrak{F} \cap R^3[0] \approx \ell$, $\partial \mathfrak{F} \cap R^3[2] \approx -\ell' \circ \mathbb{O} \circ \dots \circ \mathbb{O}$ and $\partial F_i \cap R^3[0] \approx k_i$, $\partial F_i \cap R^3[2] \approx -k'_i \circ \mathbb{O} \circ \dots \circ \mathbb{O}$ for each $i = 1, \dots, n$, where the numbers of Hopf link coincide with those of $S(\mathfrak{A}), S(A_i)$ respectively.
- (iii) There are mutually disjoint disks $\mathfrak{B}$ in $R^3[1]$ such that $\mathfrak{B}[0]$ are the disks of fission of ℓ and

$$\mathfrak{F} - (\mathfrak{B}[0] \times [0, 2]) = (\ell - \mathfrak{B}[0]) \times [0, 2],$$

where $X[i]$ means $X \times (-\infty, \infty) \cap R^3[i]$.

By Lemmas 1.2 and 2.2, we obtain the following.

LEMMA 2.3. *For a link $\ell = k_1 \cup \dots \cup k_n$, ℓ is homotopic to zero if and only if there is a non-singular locally flat proper surface $\mathfrak{F}_0 = F_1 \cup \dots \cup F_n$ of genus zero in $R^3[0, 2]$ with $\partial \mathfrak{F}_0 \cap R^3[0] \approx \ell$, $\partial F_i \cap R^3[0] \approx k_i$ and $\partial \mathfrak{F}_0 \cap R^3[2]$, $\partial F_i \cap R^3[2]$ consist of some Hopf links respectively which are splitted to each other.*

PROOF of THEOREM 2.1. First of all assume that ℓ is homotopic to ℓ' . Then there is a surface $\mathfrak{F}$ satisfying Lemma 2.2 for $\ell \subset R^3[0]$, $\ell' \subset R^3[2]$. Since ℓ' and $L_H = \partial \mathfrak{F} \cap R^3[2] - \ell'$ are splitted, there is a 3-ball E in $R^3[2]$ such that $E \supset \ell'$ and $E \cap L_H = \emptyset$.

Take a proper disk D in E such that $D \cap (\ell' \cup \mathfrak{B}[2]) = \emptyset$. Let L be a link in a connected component E_0 of $E - D$ which does not contain ℓ' such that L is symmetric to ℓ' with respect to D in E . Then L is ambient isotopic to $-\ell'$. Since $\ell' \cap E_0 = \emptyset$, ℓ' and L are splitted to each other.

Perform a fusion between ℓ' and L such that $\ell' \# L$ is symmetric with respect to D and the disks $\mathfrak{C}$ of fusion between ℓ' and L does not intersect with $\mathfrak{B}[2]$. As $E_0 \cap (\ell' \cup L_H \cup \mathfrak{B}[2]) = \emptyset$, $E_0[0]$ does not contain ℓ . So ℓ and $L[0]$ are splitted to each other. Hence $\ell \# L[0]$ can be defined by $\mathfrak{C}[0]$. Since $F \cap (L \times [0, 2]) = \emptyset$, $\mathfrak{F}_0 = (\mathfrak{F} \cup L \times [0, 2] - ((\ell' \cup L) \cap \partial \mathfrak{C}) \times [0, 2]) \cup \partial \mathfrak{C} - \ell' - L \times [0, 2]$ is a non-singular locally flat proper surface of genus zero in $R^3[0, 2]$ with $\partial \mathfrak{F}_0 \cap R^3[0] \approx \ell \# L[0]$ and $\partial \mathfrak{F}_0 \cap R^3[2] \approx (\ell' \# L) \circ L_H$. As $\ell' \# L$ is symmetric with respect to D in E , $\ell' \# L$ is cobordant to zero by Lemma 2 in [6]. Hence $\ell \# L[0]$ is homotopic to zero and as $L \approx -\ell'$, we obtain that $\ell \# (-\ell')$ is homotopic to zero by Lemma 2.3.

Conversely assume that there are disks $\mathfrak{B}$ of fusion between ℓ and $-\ell'$ in $R^3[0]$ such that $\ell \# (-\ell')$ is homotopic to zero. Then $\ell \# (-\ell') - \partial \mathfrak{B} = \ell \circ (-\ell')$.

As $\ell \# (-\ell')$ is homotopic to zero, there is a surface $\mathfrak{F}$ satisfying Lemma 2.3. Hence $\mathfrak{U} = \mathfrak{F} \cup \mathfrak{B}$ is a non-singular surface of genus zero in $R^3 [0, 2]$ with $\partial \mathfrak{U} \cap R^3 [0] = \ell \circ (-\ell')$ and $\partial \mathfrak{U} \cap R^3 [2]$ consists of completely splitted Hopf links. Therefore we see that $\ell \underset{E}{\sim} \ell'$, namely ℓ is homotopic to ℓ' by Lemma 1.2.

QUESTION 2.4. Is Theorem 2.1 true if “homotopic” replaces by “isotopic” or “cobordant”?

REMARK 2.5. The followings are true with respect to cobordism.

- (i) If $\ell \# (-\ell') \sim 0$ for a fusion between ℓ and $-\ell'$, then $\ell \sim \ell'$.
- (ii) If $\ell \sim \ell'$, there are a link L in R^3 and the disks $\mathfrak{B}, \mathfrak{B}'$ of fusion between ℓ and $-L, L$ and $-\ell'$ such that $\ell \underset{\mathfrak{B}}{\#} (-L) \sim 0$ and $L \underset{\mathfrak{B}'}{\#} (-\ell') \sim 0$.

THEOREM 2.6. Let ℓ_i and ℓ_{i+2} be homotopic links with n -component for $i = 1, 2$. Then there are fusions between ℓ_1 and ℓ_2, ℓ_3 and ℓ_4 such that $\ell_1 \# \ell_2$ is homotopic to $\ell_3 \# \ell_4$.

PROOF. As ℓ_i and ℓ_{i+2} are homotopic, there are surfaces $\mathfrak{F}_i$ and disks $\mathfrak{B}_i$ satisfying Lemma 2.2 for $i = 1, 2$, where $\ell_1, \ell_2 \subset R^3 [0]$ and $\ell_3, \ell_4 \subset R^3 [2]$. There is a 3-ball E in $R^3 [2]$ with $E \cap \partial(\mathfrak{F}_1 \cup \mathfrak{F}_2) = (-\ell_3) \cup (-\ell_4)$. Let $\mathfrak{B}$ be the disks of fusion between $-\ell_3$ and $-\ell_4$ in E such that $\mathfrak{B} \cap (\mathfrak{B}_1 [2] \cup \mathfrak{B}_2 [2]) = \emptyset$. Then $\mathfrak{F} = (\mathfrak{F}_1 \cup \mathfrak{F}_2 - \partial \mathfrak{B} \cap (\mathfrak{F}_1 \cup \mathfrak{F}_2) \times [0, 2]) \cup (\partial \mathfrak{B} - \mathfrak{F}_1 - \mathfrak{F}_2) \times [0, 2]$ is a non-singular surface of genus zero in $R^3 [0, 2]$ with $\partial \mathfrak{F} \cap R^3 [0] = \ell_1 \# \ell_2$ and $\partial \mathfrak{F} \cap R^3 [2] = (-\ell_3) \# (-\ell_4) \circ \mathbb{O} \circ \dots \circ \mathbb{O}$ by Lemma 2.2 (iii). Therefore $\ell_1 \# \ell_2$ is homotopic to $\ell_3 \# \ell_4$.

By using Lemma 1.2, the followings are obvious.

THEOREM 2.7. Let ℓ be a link homotopic to zero. For a link ℓ' , there is a fusion between ℓ and ℓ' such that $\ell \# \ell'$ is homotopic to zero, ℓ' is also homotopic to zero.

THEOREM 2.8. Let ℓ and ℓ' be homotopic links and ℓ_0 homotopic to zero. Then for any fusion between ℓ and ℓ_0, ℓ' and $\ell_0, \ell \# \ell_0$ is homotopic to $\ell' \# \ell_0$.

COROLLARY 2.9. Let ℓ, ℓ' be homotopic to zero. Then for any fusion between ℓ and $\ell', \ell \# \ell'$ is homotopic to zero.

§3. Homotopy of links in solid tori.

In this section, we consider the links in solid tori in R^3 and prove Theorems 3.1

and 3.4 which are the extension of Theorem 1.3.

Let $\mathfrak{B}^* = V_1^* \cup \dots \cup V_n^*$ be mutually disjoint n solid tori in R^3 such that $\Gamma^* = c_1^* \cup \dots \cup c_n^*$ is a trivial link for a core c_i^* of V_i^* , $i = 1, \dots, n$ and f an orientation preserving homeomorphism of $\mathfrak{B}^*$ onto $\mathfrak{B} = V_1 \cup \dots \cup V_n$ such that $f|_{V_i^*} : V_i^* \rightarrow V_i$ is faithful for $i = 1, \dots, n$. For a link $\ell^* = \ell_1^* \cup \dots \cup \ell_n^*$ in $\mathfrak{B}^*$ (or ℓ_i^* in V_i^*), we denote $f(\ell^*)$ (or $f(\ell_i^*)$) by ℓ (or ℓ_i).

In this section, we consider the cases that Γ is a boundary link and that Γ is semi-cobordant to zero and prove Theorems 3.1 and 3.4.

THEOREM 3.1. *With the above situation, if $\Gamma = f(\Gamma^*)$ is a boundary link, ℓ is homotopic to ℓ^* .*

To prove Theorem 3.1, we perform the similar discussion to that of Theorem 1.3.

Lemma 3.2 is an extension of Lemma 1.1 which can be obtained by the same proof with that of Lemma 1.1.

LEMMA 3.2. ([14]). *Let Γ , Γ^* , ℓ and ℓ^* be those of the above. If $\Gamma \sim \Gamma^*$, then $\ell \sim \ell^*$.*

PROOF OF THEOREM 3.1. If there is a link ℓ_i^* in V_i^* such that $o(\ell_i^*) = 0$ for some i , $\ell = \ell_1 \cup \dots \cup \ell_n \approx (\ell - \ell_i) \cup \ell_i \approx (\ell - \ell_i) \cup \ell_i^*$. Moreover as Γ is a boundary link, $\Gamma - c_i$ is also a boundary link. Hence to prove Theorem 3.1, it is sufficient to do that $\ell - \ell_i$ is homotopic to $\ell^* - \ell_i^*$. So we assume that $p_i = o(\ell_i^*) > 0$ for each $i = 1, \dots, n$.

Take mutually disjoint n solid tori $\mathfrak{U} = U_1 \cup \dots \cup U_n$ in R^3 with cores $-\Gamma = (-c_1) \cup \dots \cup (-c_n)$ such that $\mathfrak{U}$ is splitted from $\mathfrak{B} = V_1 \cup \dots \cup V_n$. Let L_i be a non-twisted parallel link with p_i -component in U_i such that $I(\ell_i, M_{V_i}) = -I(L_i, M_{U_i})$, see the proof of Theorem 1.3, and $L = L_1 \cup \dots \cup L_n$. Since $\mathfrak{B}$ and $\mathfrak{U}$ are splitted, $\Gamma \cup (-\Gamma) \approx \Gamma \circ (-\Gamma)$ and $\ell \cup L \approx \ell \circ L$. Hence there are the disks $\mathfrak{B}$ of fusion between Γ and $-\Gamma$ such that $\Gamma \#_{\mathfrak{B}} (-\Gamma) \sim 0 (\approx \Gamma^* \# (-\Gamma^*))$ by Lemma 2 in [6]. By $\mathfrak{B}$, $\mathfrak{U}$ and $\mathfrak{B}$, we can obtain mutually disjoint n solid tori $\mathfrak{B}_0 = V_{10} \cup \dots \cup V_{n0}$ with cores $\Gamma \#_{\mathfrak{B}} (-\Gamma)$ in R^3 such that $\mathfrak{B}_0$ contain a link ℓ_0 obtained by a fusion of $\ell \cup L$, where the constructions of $\mathfrak{B}_0$, ℓ_0 are the same to those in the proof of Theorem 1.3. Hence ℓ_0 is cobordant to ℓ_0^* by Lemma 3.2 and $\ell_0^* \approx \ell^*$. Moreover as $-\Gamma$ is a boundary link and L_i is non-twisted parallel to $-c_i$ for each i , L is also a boundary link. So L is homotopic to zero by [1], [7], [8] and [15]. Hence ℓ is E -cobordant to ℓ^* , namely ℓ is homotopic to ℓ^* by Lemma 1.2.

By Corollary 1.3.1 in [13], we obtain the following.

LEMMA 3.3. *Let Γ, Γ^* be those of Theorem 3.1. If $\Gamma = c_1 \cup \dots \cup c_n \underset{\text{semi}}{\sim} \Gamma^* (\approx 0)$, then $\Gamma \sim c_1 \circ \dots \circ c_n$.*

By Lemma 3.3, we obtain the following.

THEOREM 3.4. *Let $\Gamma, \Gamma^*, \ell = \ell_1 \cup \dots \cup \ell_n$ and $\ell^* = \ell_1^* \cup \dots \cup \ell_n^*$ be those of Theorem 3.1. If $\Gamma \underset{\text{semi}}{\sim} \Gamma^*$, then ℓ is homotopic to ℓ^* .*

PROOF. As $\Gamma \underset{\text{semi}}{\sim} \Gamma^*$, $\ell = \ell_1 \cup \dots \cup \ell_n \sim \ell_1 \circ \dots \circ \ell_n$ by Lemmas 3.2, 3.3 and each ℓ_i is homotopic to ℓ_i^* by Theorem 1.3. So $\ell_1 \circ \dots \circ \ell_n$ is homotopic to $\ell^* = \ell_1^* \circ \dots \circ \ell_n^*$. Hence ℓ is homotopic to ℓ^* .

As an extension of Theorem 3.4, we consider the following case.

Let $\mathfrak{B} = V_1 \cup \dots \cup V_n$, $\mathfrak{B}' = V'_1 \cup \dots \cup V'_n$ be mutually disjoint solid tori and g an orientation preserving homeomorphism of $\mathfrak{B}$ onto $\mathfrak{B}'$ such that $g|_{V_i}: V_i \rightarrow V'_i$ is faithful. Let c_i, c'_i be cores of V_i, V'_i respectively and $\Gamma = c_1 \cup \dots \cup c_n$, $\Gamma' = c'_1 \cup \dots \cup c'_n$. For a link ℓ in $\mathfrak{B}$, we denote $g(\ell)$ by ℓ' .

THEOREM 3.5. *If $\Gamma \underset{\text{semi}}{\sim} \Gamma'$, then ℓ is homotopic to ℓ' .*

PROOF. Let $L = C_1 \circ \dots \circ C_n$ for $C_i = c_i \# (-c'_i)$ in $R^3 - \mathfrak{B}'$ which is splitted from Γ' . Perform a fusion between Γ' and L and we obtain a link $\Gamma'' = \Gamma' \# L$. We can construct mutually disjoint solid tori $\mathfrak{B}'' = V''_1 \cup \dots \cup V''_n$ with core Γ'' and a link ℓ'' in $\mathfrak{B}''$ such that $g'(\ell) = \ell''$ for a faithful homeomorphism g' of $\mathfrak{B}$ onto $\mathfrak{B}''$ by the same proof to that of Theorem 1.3. As $\Gamma \sim \Gamma''$ by Theorem 1.5 in [13], $\ell \sim \ell''$ by Lemma 3.2. Moreover as Γ'' is obtained by a fusion between Γ and $L = C_1 \circ \dots \circ C_n$, ℓ'' is E -cobordant to ℓ' by the same proof to that of Theorem 1.3. Hence ℓ is homotopic to ℓ' .

REMARK 3.6. In Theorem 3.5, we can not replace “semi-cobordant” by “homotopic”. Namely there are links Γ, Γ^*, ℓ and ℓ^* such that Γ is homotopic to Γ^* but ℓ is not homotopic to ℓ^* , for example see Figures 3 and 4 in [7].

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