Kobe University Repository : Kernel

PDF issue: 2026-11-25

On generating functions of generalized Bessel polynomials

Chongdar, A. K.

(Citation)
Kobe Journal of Mathematics,5(1):151-154

(Issue Date)
1988-06

(Resource Type)
journal article

(Version)
Version of Record

(JaLCDOI)
https://doi.org/10.24546/E0034685

(URL)
https://hdl.handle.net/20.500.14094/E0034685

KOBE UNIVERSITY

CHONGDAR, A. K.
KOBE J. MATH.,
5(1988) 151–154

ON GENERATING FUNCTIONS OF GENERALIZED BESSEL POLYNOMIALS

By A. K. CHONGDAR

(Received March 11, 1987)

ABSTRACT

In this note the author has derived two theorems on bilateral and mixed trilateral generating functions of generalized Bessel polynomials from the Lie group view point. An application of the theorem on bilateral generating functions is also given here.

1. Introduction: The Bessel polynomials as introduced by Krall and Frink [6] are defined by,

$$y_n^{\alpha}(x) = {}_2F_0(-n,\ n+\alpha-1;\ -;\ -x/\beta).$$

Utilizing group-theoretic method [7] a large number of generating functions of Bessel polynomials have been derived by many researchers. Among their works, [2, 3, 4, 5] may be mentioned. The object of the present note is to derive the following theorem on generating functions of Bessel polynomials from the Lie group view point.

THEOREM 1. *If there exists a generating relation of the form*

$$G(x, t) = \sum_{n=0}^{\infty} a_n y_n^{(\alpha)}(x) t^n \tag{1.1}$$

then

$$\sum_{n=0}^{\infty} \sigma_n(x, t) z^n = (1-xz)^{1-\alpha} \exp(\beta z)\, G\left(\frac{x}{1-xz},\ \frac{tz}{1-xz}\right) \tag{1.2}$$

where

$$\sigma_n(x, t) = \sum_{p=0}^{n} a_p \frac{\beta^{n-p}}{(n-p)!} y_n^{(\alpha-n+p)}(x)\, t^p. \tag{1.3}$$

THEOREM 2, *If there exists a generating relation of the form*

$$G(x, y, t) = \sum_{n=0}^{\infty} a_n y_n^{(\alpha)}(x) g_n(y) t^n \tag{1.4}$$

then

$$(1.5)\quad \sum_{n=0}^{\infty} \sigma_n(x, y, t)z^n = (1-xz)^{1-\alpha}\exp(\beta z)G\left(\frac{x}{1-xz}, y, \frac{tz}{1-xz}\right)$$

where

$$(1.6)\quad \sigma_n(x, y, t) = \sum_{p=0}^{n} a_p \frac{\beta^{n-p}}{(n-p)!} y_n^{(\alpha-n+p)}(x)g_p(y)t^p .$$

From the above Theorems it is obvious that whenever one knows a unilateral or a bilateral generating relation of the form (1.1) or (1.4) involving generalized Bessel polynomials then the corresponpding bilateral or mixed trilateral generating relation can at once be written down from (1.2) and (1.5). So a large number of bilateral or trilateral generating relations can be derived by attributing different values to a_n.

PROOF OF THEOREM 1. For the Bessel polynomials, we first consider the following partial differential operator [3]:

$$\mathcal{R} = x^2 y^{-1} z \frac{\partial}{\partial x} + xz \frac{\partial}{\partial y} + xy^{-1}z^2 \frac{\partial}{\partial z} + y^{-1}z(\beta - x)$$

such that

$$(1.7)\quad \mathcal{R}(y_n^{(\alpha)}(x)y^{\alpha}z^n) = \beta y_{n+1}^{(\alpha-1)}(x)y^{\alpha-1}z^{n+1}.$$

The extended form of the group generated by $\mathcal{R}$ is

$$(1.8)\quad \exp(\omega\mathcal{R})f(x, y, z) = (1-\omega xy^{-1}z)\exp(\beta\omega y^{-1}z) \times f\left(\frac{x}{1-\omega xy^{-1}z}, \frac{y}{1-\omega xy^{-1}z}, \frac{z}{1-\omega xy^{-1}z}\right).$$

Now let us consider the following generating relation:

$$(1.9)\quad G(x, t) = \sum_{n=0}^{\infty} a_n y_n^{(\alpha)}(x)t^n.$$

Replacing t by tz and then multiplying both sides of (1.9) by y^{α}, we get

$$(1.10)\quad y^{\alpha}G(x, tz) = \sum_{n=0}^{\infty} a_n t^n (y_n^{(\alpha)}(x)y^{\alpha}z^n).$$

Operating both sides of the relation (1.10) by $(\exp(\omega\mathcal{R}))$ we get

$$\exp(\omega R)(y^{\alpha}G(x,tz)) = \exp(\omega R)\sum_{n=0}^{\infty} a_n t^n (y_n^{(\alpha)}(x)y^{\alpha}z^n). \tag{1.11}$$

The left member of (1.11) is

$$(1-\omega x y^{-1}z)^{1-\alpha}\exp(\beta\omega y^{-1}z)y^{\alpha}G\left(\frac{x}{1-\omega x y^{-1}z}, \frac{y}{1-\omega x y^{-1}z}\right). \tag{1.12}$$

The right member of (1.11) is

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n t^n \frac{\omega^m}{m!} R^m (y_n^{(\alpha)}(x)y^{\alpha}z^n) \qquad (1.13)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n t^n \frac{\omega^m}{m!}\beta^m y_{n+m}^{(\alpha-m)}(x)y^{\alpha-m}z^{n+m}\\
&= y^{\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n t^n \frac{(\omega\beta y^{-1})^m}{m!} z^{n+m} y_{n+m}^{(\alpha-m)}(x).
\end{aligned}$$

equating the relations (1.12) and (1.13) and then putting we get

$$\begin{aligned}
&(1-xz)^{1-\alpha}\exp(\beta z)G\left(\frac{x}{1-xz}, \frac{tz}{1-xz}\right)\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{n-m} t^{n-m}\frac{\beta^m}{m!} z^n y_n^{(\alpha-m)}(x)\\
&= \sum_{n=0}^{\infty} z^n \sum_{m=0}^{n} a_{n-m} t^{n-m}\frac{\beta^m}{m!} y_n^{(\alpha-m)}(x)\\
&= \sum_{n=0}^{\infty}\sigma_n(x,t)z^n
\end{aligned}$$

where

$$\sigma_n(x,t) = \sum_{p=0}^{n} a_p \frac{\beta^{n-p}}{(n-p)!} y_n^{(\alpha-n+p)}(x)t^p.$$

this completes the proof of the theorem 1.

As the proof of the theorem 2 is exactly similar to that of theorem 1, we omit it here.

3. Application: In this section we shall give an application of theorem 1. From [1], we notice that

$$[\frac{1}{2}-\frac{1}{2}(1-2xt)^{\frac{1}{2}}]^{2-\alpha}(1-2xt)^{-\frac{1}{2}}\exp[\frac{\beta}{2x}\left\{1-(1-2xt)^{\frac{1}{2}}\right\}]$$

$$=\sum_{n=0}^{\infty}\frac{\beta^n}{2^n n!}y_n^{(\alpha)}(x)t^n.$$

As an application of theorem 1, if we take

$$a_n=\frac{\beta^n}{2^n n!},$$

We get from (3.1),

$$(3.2)\quad G(x,t)=[\frac{1}{2}-\frac{1}{2}(1-2xt)^{\frac{1}{2}}]^{2-\alpha}(1-2xt)^{-\frac{1}{2}}\exp[\frac{\beta}{2x}\left\{1-(1-2xt)^{\frac{1}{2}}\right\}].$$

Therefore from our theorem 1, we at once get

$$(3.3)\qquad (\frac{\rho_2}{2})^{2-\alpha}\rho_1^{-1}\exp[\beta(z+\frac{\rho_2}{2x})]=\sum_{n=0}^{\infty}\sigma_n(x,t)z^n$$

where

$$\sigma_n(x,t)=\sum_{p=0}^{n}\frac{(\frac{t}{2})^p}{p!}\frac{\beta^n}{(n-p)!}y_n^{(\alpha-n+p)}(x),$$

$$\rho_1^2=1-2xz(1+t)+x^2z^2,$$

$$\rho_2=1-xz-\rho_1.$$

References

[1] Burchnall, J. L.: The Bessel polynomials, Canad. Jour. Math. 3 (1951), 62–68.

[2] Chen, M. P. and Feng, C.C.: Group-theoretic origins of certain generating functions of generalized Bessel polynomials, Tamkang Jour. Math. 6 (1975), 87–93.

[3] Chongdar, A. K.: Some generating functions involving generalized Bessel polynomials, Bull. Cal. Math. Soc. 77 (6) (1985), 330–339.

[4] Chongdar, A. K.: Some generating functions of generalized Bessel polynomials, Bull. Cal. Math. Soc. 76 (6) (1984), 362–369.

[5] Guha Thakurta, B. K. and A. K. Chongdar: Some generating functions of modified Bessel polynomials by Lie algebraic method, Bull. Inst. Math. Acad. Sinica, 14 (2), (1986), 215–221.

[6] Krall, H. L. and Frink, O.: A new class of orthogonal polynomials: The Bessel polynomials, Trans. Amer. Math. Soc. 65 (1949), 100–115.

[7] Weisner, L.: Group-theoretic origins of certain generating functions, Pacific Jour. Math. 5 (1955), 1033–1039.

Department of Mathematics
19, Rajkumar Chakraborty Sarani,
Calcutta – 700 009, INDIA.