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A NOTE ON ENDOMORPHISM SEMIRINGS OF SEMIMODULES

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Let Λ be a semiring with $1 \notin s(\Lambda)$. In this note, for each Λ-semimodule A, we construct a ring homomorphism of $K(\mathrm{End}_\Lambda(A))$ to $\mathrm{End}_{K(\Lambda)}(K(A))$. As the results we obtain two bimodule structures

$$K(\Lambda)K(A)_{K(\mathrm{End}_\Lambda(A))} \quad \text{and} \quad K(\Lambda)K(A)_{K(\mathrm{End}_\Lambda(c(A))}.$$

Furthermore in the case when Λ is the semiring N of non negative integers and Z is the ring of integres, we prove that $K(\mathrm{End}_N(A)) \cong \mathrm{End}_Z(K(A))$ for each cyclic semimodule A.

The definitions and notions used here we refer to [3], [4] and [9].

Let Λ be a semiring with $1 \notin s(\Lambda)$ and A a Λ-semimodule. Let $\Delta(A) = \{(a, a) \mid a \in A\}$, $\Delta(\Lambda) = \{(\lambda, \lambda) \mid \lambda \in \Lambda\}$, $K(A) = (A \times A)/\Delta(A)$ and $K(\Lambda) = (\Lambda \times \Lambda)/\Delta(\Lambda)$, for which in [3] we have seen that $K(\Lambda)$ is a ring with respect to

$$\begin{aligned}
&\overline{(\lambda, \nu)} + \overline{(\lambda', \nu')} = \overline{(\lambda + \lambda', \nu + \nu')} \\
&\overline{(\lambda, \nu)}\,\overline{(\lambda', \nu')} = \overline{(\lambda\lambda' + \nu\nu', \lambda\nu' + \nu\lambda')} \qquad (\lambda, \nu, \lambda', \nu' \in A)
\end{aligned}$$

and $K(A)$ is a $K(\Lambda)$-module with respect to

$$\begin{aligned}
&\overline{(a, b)} + \overline{(a', b')} = \overline{(a + a', b + b')} \\
&\overline{(\lambda, \nu)}\,\overline{(a, b)} = \overline{(\lambda a + \nu b, \nu a + \lambda b)} \qquad (a, a', b, b' \in A;\ \lambda, \nu \in \Lambda).
\end{aligned}$$

Now we define $\theta\colon K(\mathrm{End}_\Lambda(A)) \longrightarrow \mathrm{End}_{K(\Lambda)}(K(A))$ by

$$\theta\,\overline{(f, g)}\,(\overline{(a, b)}) = \overline{(f(a) + g(b), f(b) + g(a))}.$$

Then the map θ is well-defined. In fact, first if $\overline{(f, g)} = \overline{(f_1, g_1)}$, then there exist $m, n \in \mathrm{End}_\Lambda(A)$ such that

$$f + m = f_1 + n \quad \text{and} \quad g + m = g_1 + n.$$

So we have the following identities:

$$f(a) + m(a) = f_1(a) + n(a) \quad \langle 1 \rangle$$
$$g(a) + m(a) = g_1(a) + n(a) \quad \langle 2 \rangle$$
$$f(b) + m(b) = f_1(b) + n(b) \quad \langle 3 \rangle$$
$$g(b) + m(b) = g_1(b) + n(b) \quad \langle 4 \rangle$$

for any a, b in A. By $\langle 1 \rangle + \langle 4 \rangle$ and $\langle 2 \rangle + \langle 3 \rangle$

$(f(a)+g(b)) + m(a+b) = (f_1(a)+g_1(b)) + n(a+b)$ and
$(g(a) + f(b)) + m(a+b) = (g_1(a) + f_1(b)) + n(a+b)$ hold. It follows that $\overline{(f(a) + g(b), f(b) + g(a))} = \overline{(f_1(a) + g_1(b), f_1(b) + g_1(a))}$, and so $\theta(\overline{f, g}) = \theta(\overline{f_1, g_1})$.

Secondly, suppose that $(\overline{a, b}) = (\overline{a_1, b_1})$, that is, $a + c = a_1 + d$ and $b + c = b_1 + d$ for some c, d in A. Then we have

$$f(a) + f(c) = f(a_1) + f(d) \quad \langle 5 \rangle$$
$$g(a) + g(c) = g(a_1) + g(d) \quad \langle 6 \rangle$$
$$f(b) + f(c) = f(b_1) + f(d) \quad \langle 7 \rangle$$
$$g(b) + g(c) = g(b_1) + g(d). \quad \langle 8 \rangle$$

By $\langle 5 \rangle + \langle 8 \rangle$ and $\langle 6 \rangle + \langle 7 \rangle$ the following identities hold:

$$(f(a) + g(b)) + (f(c) + g(c)) = (f(a_1) + g(b_1)) + (f(d) + g(d)),$$
$$(f(b) + g(a)) + (f(c) + g(c)) = (f(b_1) + g(a_1)) + (f(d) + g(d)).$$

This implies

$$\overline{(f(a)+g(b), f(b)+g(a))} = \overline{(f(a_1)+g(b_1), f(b_1)+g(a_1))},$$

and so $\theta(\overline{f, g})(\overline{a, b}) = \theta(\overline{f, g})(\overline{a_1, b_1})$.

Finally, we claim that $\theta(\overline{f, g}) \in \mathrm{End}_{K(\Lambda)}(K(A))$, since

$$\theta(\overline{f, g})((\overline{a, b}) + (\overline{c, d})) = \theta(\overline{f, g})(\overline{a + c, b + d})$$
$$= \overline{(f(a)+f(c)+g(b)+g(d), f(b)+f(d)+g(a)+g(c))}$$
$$= \overline{(f(a)+g(b), f(b)+g(a))} + \overline{(f(a)+g(d), f(d)+g(c))}$$
$$= \theta(\overline{f, g})(\overline{a, b}) + \theta(\overline{f, g})(\overline{c, d}).$$

On the other hand, for $(\overline{\lambda, \mu}) \in K(\Lambda)$, $(\overline{a, b}) \in K(A)$,

$$\theta(\overline{f, g})((\overline{\lambda, \mu})(\overline{a, b})) = \theta(\overline{f, g})(\overline{\lambda a + \mu b, \mu a + \lambda b})$$
$$= \overline{(\lambda f(a) + \mu f(b) + \mu g(a) + \lambda g(b), \mu f(a) + \lambda f(b) + \lambda g(a) + \mu g(b))}$$
$$= \overline{(\lambda(f(a)+g(b)) + \mu(f(b)+g(a)), \mu(f(a)+g(b)) + \lambda(f(b)+g(a))}$$

$= \overline{(\lambda, \mu)}\,\overline{(f(a)+g(b),\ f(b)+g(a))}$

$= \overline{(\lambda, \mu)}\,(\theta\,\overline{(f, g)}\,(\overline{(a, b)})).$

So far we have shown that θ is well-defined.

THEOREM 1. *Let* Λ *be a semiring with* $1 \notin s(\Lambda)$. *Then the map* θ *defined above is a ring homomorphism of* $K(\mathrm{End}_\Lambda(A))$ *into* $\mathrm{End}_{K(\Lambda)}(K(A))$.

PROOF. Since

$\theta\,(\overline{(f, g)} + \overline{(f_1, g_1)})\,\overline{(a, b)}$

$= \theta\overline{(f+f_1,\ g+g_1)}\,\overline{(a, b)}$

$= \overline{((f+f_1)(a)+(g+g_1)(b),\ (f+f_1)(b)+(g+g_1)(a))}$

$= \overline{((f(a)+g(b))+(f_1(a)+g_1(b)),\ (g(a)+f(b))+(g_1(a)+f_1(b)))}$

$= \overline{(f(a)+g(b),\ f(b)+g(a))} + \overline{(f_1(a)+g_1(b),\ f_1(b)+g_1(a))}$

$= \theta\,\overline{(f, g)}\,\overline{(a, b)} + \theta\overline{(f_1, g_1)}\,\overline{(a, b)}$, we have

$\theta(\overline{(f, g)} + \overline{(f_1, g_1)}) = \theta\overline{(f, g)} + \theta\overline{(f_1, g_1)}.$

On the other hand

$\theta(\overline{(f, g)}\,\overline{(f_1, g_1)})\,\overline{(a, b)}$

$= \theta\overline{(ff_1+gg_1,\ fg_1+gf_1)}\,\overline{(a, b)}$

$= \overline{(ff_1(a)+gg_1(a)+fg_1(b)+gf_1(b),\ ff_1(b)+gg_1(b)+fg_1(a)+gf_1(a))}$

$= \overline{(f(f_1(a)+g_1(b))+g(g_1(a)+f_1(b)),\ f(f_1(b)+g_1(a))+g(g_1(b)+f_1(a))}$

$= \theta\overline{(f, g)}\,(\theta\overline{(f_1, g_1)}\,\overline{(a, b)})$, and so

$\theta(\overline{(f, g)}\,\overline{(f_1, g_1)}) = \theta\,\overline{(f, g)}\,\theta\overline{(f_1, g_1)}.$

In addition, $\theta\,\overline{(1_A, 0)}\,\overline{(a, b)} = \overline{(1_A a + 0b,\ 0a + 1_A b)} = \overline{(a, b)}$. That is, $\theta\overline{(1, 0)} = 1_{K(A)}$, where $\overline{(1_A, 0)}$ and $1_{K(A)}$ are identities for $K(\mathrm{End}_\Lambda(A))$ and $\mathrm{End}_{K(\Lambda)}(K(A))$, respectively. Clearly $\theta\overline{(0, 0)} = 0$. So we have proved that θ is a ring homomorphism. This completes the proof.

Let us now consider the ring homomorphism

$$\theta : K(\mathrm{End}_\Lambda(A)) \longrightarrow \mathrm{End}_{K(\Lambda)}(K(A)).$$

Then the map θ is a homomorphism of $K(\Lambda)$-modules. Thus we have the following exact sequence of $K(\Lambda)$-modules:

$$0 \longrightarrow \mathrm{Ker}(\theta) \rightarrowtail K(\mathrm{End}_\Lambda(A)) \xrightarrow{\theta} \mathrm{End}_{K(\Lambda)}(K(A)), \text{ where}$$

$\mathrm{Ker}(\theta) = \{ \overline{(f,g)} \mid \theta\overline{(f,g)} = 0 \}$ is an ideal of $K(\mathrm{End}_\Lambda(A))$ in the sence of the ring theory.

The ideal $\mathrm{Ker}(\theta)$ is characterized as follows:

LEMMA $\mathrm{Ker}(\theta) \ni \overline{(f,g)} \Longleftrightarrow$ *For each element* $a \in A$, *there exists an element* $u \in A$ *such taht* $f(a)+u = g(a)+u$.

PROOF [$\Longrightarrow$]: Suppose that $\overline{(f,g)} \in \mathrm{Ker}(\theta)$. Since $\theta\overline{(f,g)} = 0$, for every element $\overline{(a,0)} \in K(A)$, $\overline{(0,0)} = \theta\overline{(f,g)}\,\overline{(a,0)} = \overline{(f(a)+g(0), g(a)+f(0))} = \overline{(f(a), g(a))}$. So there exists $u \in A$ such that $f(a)+u = g(a)+u$. [$\Longleftarrow$]: For $\overline{(a,b)} \in K(A)$, by assumption there exist u, v in A such that $f(a)+u = g(a)+u$ and $f(b)+v = g(b)+v$, and so $f(a)+g(b)+(u+v) = f(b)+g(a)+(u+v)$. It follows $\overline{(f(a)+g(b), f(b)+g(a))} = \overline{(0,0)}$. Therefore $\overline{(f,g)} \in \mathrm{Ker}(\theta)$.

COROLLARY 1 *If* A *is cacellative, then* $\mathrm{Ker}(\theta) = 0$.

The proof follows from the lemma.

COROLLARY 2 *Let* Λ *be a semiring with* $1 \notin s(\Lambda)$. *Then for each* Λ*-semimodule* A, *there exists a ring monomorphism* $\bar{\theta}$:

$$K(\mathrm{End}_\Lambda(c(A))) \longrightarrow \mathrm{End}_{K(\Lambda)}(K(A)).$$

PROOF By Theorem 1 and Corollary 1 there exists a monomorphism θ: $K(\mathrm{End}_\Lambda(c(A))) \longrightarrow \mathrm{End}_{K(\Lambda)}(K(c(A)))$. But $K(c(A)) \cong K(A)$ (see [3]), so $\mathrm{End}_{K(\Lambda)}(K(c(A))) \cong \mathrm{End}_{K(\Lambda)}(K(A))$. It means that θ deduces a monomorphism $\bar{\theta}$: $K(\mathrm{End}_\Lambda(c(A))) \longrightarrow \mathrm{End}_{K(\Lambda)}(K(A))$.

By the theorem and corollaries, since $\mathrm{End}_{K(\Lambda)}(K(A))$ is a subring of $\mathrm{End}_Z(K(A))$, we immediately obtain two bimodule structures

$$_{K(\Lambda)}K(A)_{K(\mathrm{End}_\Lambda(A))} \quad \text{and} \quad _{K(\Lambda)}K(A)_{K(\mathrm{End}_\Lambda(c(A)))}.$$

Let us now consider the following diagram in the case of $\Lambda = N$:

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\mathrm{End}} & \mathbf{Sring} \\ \downarrow K & & \downarrow K \\ \mathbf{M} & \xrightarrow{\mathrm{End}} & \mathbf{Ring} \end{array}$$

where **S** is the category of semimodules, **M** is the category of modules, **Sring** is the category of semirings and **Ring** is the category of rings. Unfortunately the cor-

respondence End is not a functor and the diagram is not commutative in general. That is, for each semimodule A, $K(\mathrm{End}_N(A))$ is not necessarily isomorphic to $\mathrm{End}_Z(K(A))$. However, for cyclic semimodule we have the following theorem.

THEOREM 2 *Let A be a cyclic semimodule generated by a. Then*

$$K(\mathrm{End}_N(A)) \cong \mathrm{End}_Z(K(A)).$$

PROOF By the theorem 1.10 in [9], A is isomorphic to $N(a)$ or $S_{n,q}(a)$ with $0 \leqslant q < n$. We discuss with the following two cases.

CASE 1: if $A = N(a)$, then $K(A) \cong Z(a) \cong Z$, and so $\mathrm{End}_Z(K(A)) \cong \mathrm{End}_Z(Z) \cong Z$. On the other hand, the fact $\mathrm{Hom}_N(N(a), N(b)) \cong N(b)$ implies that $\mathrm{End}_N(A) \cong N(a)$ and $K(N(a)) \cong Z(a) \cong Z$.

CASE 2: if $A = S_{n,q}(a)$, then $K(A) \cong Z_{n-q}$ by theorem (2.4) in [9], and $\mathrm{End}(K(A)) \cong \mathrm{Hom}(Z_{n-q}, Z_{n-q}) \cong Z_{n-q}$. The fact $\mathrm{Hom}_N(S_{n,q}(a), S_{m,p}(b)) \cong {}_{n,q}(S_{m,p}(b))$ (Proposition 2.6 [9]) implies that $\mathrm{End}_N(S_{n,q}(a)) \cong {}_{n,q}S_{n,q}(a) = S_{n,q}(a)$. So $K(\mathrm{End}_N(A)) \cong K(S_{n,q}(a)) \cong Z_{n-q} \cong \mathrm{End}_Z(K(A))$. Thus we complete the proof.

REMARK. It should be noted that for Λ-semimodule A $\mathrm{End}_\Lambda(A) \subset \mathrm{End}_N(A)$ and $\mathrm{End}_{K(\Lambda)}(K(A)) \subset \mathrm{End}_Z(K(A))$ in general.

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