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HILBERT ALGEBRAS OF FRACTIONS AND MAXIMAL HILBERT ALGEBRAS OF QUOTIENTS

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The concept of *maximal lattice of quotients* for a distributive lattice was defined by J. Schmid in [11, 12] (taking as a guide — line the elegant construction of *complete ring of quotients* by partial morphisms introduced by G. Findlay and J. Lambek (see [8, p. 36])). The central role in the construction of the maximal lattice of quotients for a distributive lattice due to J. Schmid in [11, 12] is played by the concept of *multiplier* for a distributive lattice defined by W.H. Cornish in [4] (for a study of multiplier extensions of implicative BCK-algebras see [5]).

The scope of this paper is to define (taking as a guide-line the case of distributive lattices) the notions of *Hilbert algebra of fraction* and *maximal Hilbert algebra of quotients* for a Hilbert algebra (in this paper we shall consider only bounded Hilbert algebras). For some informal explanations of notion of „fraction“ see [8, p. 37] for the case of rings.

The paper is organized as follows. In section 1 we recall the basic definitions.

In section 2 we put in evidence many rules of calculus in Hilbert algebras which we need in the rest of paper (especially (c_{16}) and (c_{21})). In section 3 we define the notion of *multiplier* for a Hilbert algebra (also we put in evidence many results which we need in the rest of paper, especially in the section 4).

In section 4 we define the notions of *Hilbert algebra of fraction* and *maximal Hilbert algebra of quotients* for a Hilbert algebra.

In the last part of this paper is proved the existence of the *maximal Hilbert algebra of quotients* for a Hilbert algebra (theorem 4.5.).

I would like to express my gratitude for the guidance given by the referee in the elaboration of this paper.

1. Definitions and preliminaries

DEFINITION 1.1. A *Hilbert algebra* is a triple $(A, \rightarrow, 1)$, where A is a non-empty set, $\rightarrow$ is a binary operation on A , $1 \in A$ is an element such that the next axioms is verified for every $x, y, z \in A$:

$$(a_1) \quad x \rightarrow (y \rightarrow x) = 1$$

$$(a_2) \quad (x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) = 1$$

$$(a_3) \quad \text{If } x \rightarrow y = y \rightarrow x = 1, \text{ then } x = y.$$

If A is a Hilbert algebra, then the relation $x \leq y$ iff $x \rightarrow y = 1$ is a partial order on A (which will be called *natural ordering* on A); with respect to this ordering 1 is the largest element of A . A *bounded* Hilbert algebra is a Hilbert algebra with a smallest element 0 .

In this paper the symbols $\Rightarrow$ and $\Leftrightarrow$ is used for logical implication and logical equivalence.

EXAMPLES.

1. If $(A, \vee, \wedge, \neg, 0, 1)$ is a Boolean lattice, then $(A, \rightarrow, 1)$ is a bounded Hilbert algebra, where $\rightarrow$ is defined by $x \rightarrow y = (\neg x) \vee y$ for $x, y \in A$.

2. If $(A, \leq)$ is a poset, then $(A, \rightarrow, 1)$ is a Hilbert algebra, where 1 is the largest element of A and $x \rightarrow y = 1$ if $x \leq y$ and y otherwise for $x, y \in A$.

3. Let $(A, \leq)$ be a poset and $a, b \in A$; the *relative pseudocomplement* of a with respect to b , denoted by $a \rightarrow b$, is an element of A such that for every $x \in A$ the following equivalence holds: (for every $z \in A$ ($z \leq a$ and $z \leq x \Rightarrow z \leq b$)) iff $x \leq a \rightarrow b$ (the above definition is due to Katrinák in [7]). If $a \rightarrow b$ exists for every $a, b \in A$, we say that A is a *relatively pseudocomplemented poset*.

If $(A, \leq)$ is a relatively pseudocomplemented poset, then it is proved that $(A, \rightarrow, 1)$ is a Hilbert algebra ([10, p. 76]), where $1 = a \rightarrow a$ for an element $a \in A$ and the natural ordering on A coincides with the original order $\leq$.

4. If (X, τ) is a topological space, then $(\tau, \rightarrow, X)$ is a bounded Hilbert algebra, where for $D_1, D_2 \in \tau$, $D_1 \rightarrow D_2 = \text{int}((X - D_1) \cup D_2)$ and $0 = \emptyset$.

5. A *Heyting algebra* [1, p. 174] is a lattice $(A, \vee, \wedge)$ with 0 such that for every $x, y \in A$ there exists an element $x \rightarrow y \in A$ such that for $z \in A$, $x \wedge z \leq y$ iff $z \leq x \rightarrow y$. If $(A, \vee, \wedge, 0)$ is a Heyting algebra, then $(A, \rightarrow, 1)$ is a bounded Hilbert algebra, where $1 = a \rightarrow a$ for an $a \in A$.

If we consider A only with $\wedge$, we obtain the notion of *implicative semi lattice* [9].

In the rest of this paper, by A we denote a bounded Hilbert algebra.

If A_1, A_2 are Hilbert algebras, then $f: A_1 \rightarrow A_2$ is called a *morphism of Hilbert algebras* if $f(0) = 0$ and $f(x \rightarrow y) = f(x) \rightarrow f(y)$ for every $x, y \in A_1$.

We denote by $Ds(A)$ the set of all *deductive systems* of A (see [2, 3]).

$S \subseteq A$ is called *Hilbert subalgebra* of A if $0 \in S$ and $x, y \in S \Rightarrow x \rightarrow y \in S$.

2. Rules of calculus in Hilbert algebras

THEOREM 2.1. [6, p. 5]. *If $x, y, z \in A$, then the following hold:*

$$(c_1) \quad x \leq y \rightarrow x$$

- (c₂) $x \rightarrow 1 = 1$
- (c₃) $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$
- (c₄) $(x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x) = (y \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow y)$
- (c₅) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$
- (c₆) $x \leq (x \rightarrow y) \rightarrow y, ((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$
- (c₇) $1 \rightarrow x = x$
- (c₈) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$
- (c₉) If $x \leq y$, then $z \rightarrow x \leq z \rightarrow y$ and $y \rightarrow z \leq x \rightarrow z$.

If $x \in A$, then by x^* we denote the element $x \rightarrow 0$.

From theorem 2.1. we deduce immediately

THEOREM 2.2. If $x, y \in A$, then:

- (c₁₀) $0^* = 1, 1^* = 0$
- (c₁₁) $x \rightarrow y^* = y \rightarrow x^*$
- (c₁₂) $x \rightarrow x^* = x^*, x^* \rightarrow x = x^{**}$
- (c₁₃) $x \rightarrow y \leq y^* \rightarrow x^*$
- (c₁₄) If $x \leq y$ then $y^* \leq x^*$.

REMARK 2.3. If (X, τ) is a topological space and $D \in \tau$, then $D^* = \text{int}(X - D)$ and $D^{**} = \text{int}(\bar{D})$ ($\bar{D}$ = the adherence of D). If A is a Boolean lattice, then for $x \in A$, $x^* = \neg x$.

For $x, y \in A$, we denote by $x \mathbb{W} y = x^* \rightarrow y$ and $x \dot{\mathbb{W}} y = (x \rightarrow y) \rightarrow ((y \rightarrow x) \rightarrow x)$.

LEMMA 2.4. If $x, y, z \in A$, then

- (c₁₅) $x \leq x \mathbb{W} y, y \leq x \mathbb{W} y, x \mathbb{W} x = x^{**}, x \mathbb{W} 1 = 1, x \mathbb{W} x^* = 1, x \mathbb{W} (y \rightarrow z) = (x \mathbb{W} y) \rightarrow (x \mathbb{W} z)$.
- (c₁₆) $x \mathbb{W} (y \mathbb{W} z) = (x \mathbb{W} y) \mathbb{W} z = y \mathbb{W} (x \mathbb{W} z)$.

PROOF. (see also [3]). (c₁₅). We have $x \rightarrow (x^* \rightarrow y) = (x \rightarrow x^*) \rightarrow (x \rightarrow y) = x^* \rightarrow (x \rightarrow y) = (x \rightarrow 0) \rightarrow (x \rightarrow y) = x \rightarrow (0 \rightarrow y) = x \rightarrow 1 = 1$ and so $x \leq x^* \rightarrow y = x \mathbb{W} y$ (the relation $y \leq x \mathbb{W} y$ follows from (c₁)); also, $x \mathbb{W} x = x^* \rightarrow x = x^{**}$, $x \mathbb{W} 0 = x^* \rightarrow 0 = x^{**}$, $x \mathbb{W} 1 = x^* \rightarrow 1 = 1$. The relation $x \mathbb{W} (y \rightarrow z) = (x \mathbb{W} y) \rightarrow (x \mathbb{W} z)$ follows from (c₃).

(c₁₆). (after an idea of the referee). $y \leq x^* \rightarrow y \Rightarrow (x^* \rightarrow y)^* \leq y^* \Rightarrow y^* \rightarrow z \leq (x^* \rightarrow y)^* \rightarrow z$. Hence, $x^* \rightarrow (y^* \rightarrow z) \leq x^* \rightarrow ((x^* \rightarrow y)^* \rightarrow z) \leq (x^* \rightarrow y)^* \rightarrow ((x^* \rightarrow y)^* \rightarrow z)$ (since $(x^* \rightarrow y)^* \leq x^*$) $= (x^* \rightarrow y)^* \rightarrow z$ (1).

On the other hand,

$x^* \leq (x^* \rightarrow y) \rightarrow y$ (by (c₆)) $\leq y^* \rightarrow (x^* \rightarrow y)^*$ (by (c₁₃)) $\leq ((x^* \rightarrow y)^* \rightarrow z) \rightarrow$

$(y^* \rightarrow z)$ (by (c_8)).

Therefore,

$$1 = x^* \rightarrow (((x^* \rightarrow y)^* \rightarrow z) \rightarrow (y^* \rightarrow z)) = ((x^* \rightarrow y)^* \rightarrow z) \rightarrow (x^* \rightarrow (y^* \rightarrow z)).$$

Thus,

$$(x^* \rightarrow y)^* \rightarrow z \leq x^* \rightarrow (y^* \rightarrow z) \quad (2).$$

By (1) and (2), we have that $x \mathbb{W} (y \mathbb{W} z) = x^* \rightarrow (y^* \rightarrow z) = (x^* \rightarrow y)^* \rightarrow z = (x \mathbb{W} y) \mathbb{W} z$.

The relation $x \mathbb{W} (y \mathbb{W} z) = y \mathbb{W} (x \mathbb{W} z)$ follows from (c_5) .

Obs. For an another proof of (c_{16}) see [3, p. 14].

LEMMA 2.5. *If $x, y, z \in A$, then:*

$$(c_{17}) \quad x \dot{\vee} y = y \dot{\vee} x, \quad x \leq x \dot{\vee} y, \quad y \leq x \dot{\vee} y$$

$$(c_{18}) \quad x \dot{\vee} x = x, \quad x \dot{\vee} 0 = x^{**}, \quad x \dot{\vee} 1 = 1$$

$$(c_{19}) \quad x \dot{\vee} (x \rightarrow y) = 1, \quad x \dot{\vee} x^* = 1$$

$$(c_{20}) \quad (x \rightarrow y) \dot{\vee} z = x \rightarrow (y \dot{\vee} z).$$

PROOF. See [3, p. 12].

COROLLARY 2.6. *If $x, y \in A$, then:*

$$(c_{21}) \quad (x \rightarrow y)^{**} = x^{**} \rightarrow y^{**}.$$

PROOF. If in (c_{20}) we put $z = 0$, we obtain $(x \rightarrow y) \dot{\vee} 0 = x \rightarrow (y \dot{\vee} 0) \Leftrightarrow (x \rightarrow y)^{**} = x \rightarrow y^{**} \Leftrightarrow (x \rightarrow y)^{**} = x^{**} \rightarrow y^{**}$ (since $x^{**} \rightarrow y^{**} = y^* \rightarrow x^{***} = y^* \rightarrow x^* = x \rightarrow y^{**}$).

LEMMA 2.7. [2, p. 30]. *The following assertions are equivalent:*

- (i) A is Boolean lattice relative to the natural ordering.
- (ii) For every $x \in A$, $x^{**} = x$.

3. Multipliers on a Hilbert algebra

DEFINITION 3.1. $S \subseteq A$ is called a $\mathbb{W}$ -subset of A if for every $a \in A$ and $x \in S$ we have $a \mathbb{W} x \in S$.

We denote by $S(A)$ the set of all $\mathbb{W}$ -subsets of A ; clearly $Ds(A) \subseteq S(A)$.

LEMMA 3.2. *If $D \in S(A)$, then:*

- (i) $1 \in D$
- (ii) $x \in D \Rightarrow x^{**} \in D$.

PROOF. (i). If $x \in D$, since $1 \in A$, we have $1 \mathbb{W} x \in D \Leftrightarrow 0 \rightarrow x = 1 \in D$.

(ii). If $x \in D$, then $x \vee x \in D \Leftrightarrow x^{**} \in D$.

DEFINITION 3.3. By a *partial multiplier* on A we mean a map $f : D \rightarrow A$, where $D \in S(A)$, such that for every $x, y \in D$ and $a \in A$:

$$(a_4) \quad f(a \vee x) = a \vee f(x)$$

$$(a_5) \quad f(x^{**}) = f(x)$$

$$(a_6) \quad x \vee f(y) = y \vee f(x).$$

By $\text{dom}(f) \in S(A)$ we denote the domain of f ; if $\text{dom}(f) = A$, we called f *total*.

To simplify language, we will use „multiplier“ instead „partial multiplier“, using „total“ to indicate that the domain of a certain multiplier is A .

EXAMPLES.

1. The map $1 : A \rightarrow A$ defined by $1(x) = 1$ for every $x \in A$ is a total multiplier on A ; indeed, if $x, a \in A$, then $a \vee 1(x) = a^* \rightarrow 1 = 1 = 1(a \vee x)$ and $1(x^{**}) = 1 = 1(x)$. Also, if $x, y \in A$, then $x \vee 1(y) = x^* \rightarrow 1 = 1$ and $y \vee 1(x) = y^* \rightarrow 1 = 1$, hence $x \vee 1(y) = y \vee 1(x)$.

2. The map $0 : A \rightarrow A$ defined by $0(x) = x^{**}$ for every $x \in A$ is also a total multiplier on A ; indeed, if $x, a \in A$, then $0(a \vee x) = (a \vee x)^{**} = a^{***} \rightarrow x^{**} = (\text{by } (c_{21})) a^* \rightarrow x^{**} = a \vee 0(x)$ and $0(x^{**}) = x^{****} = x^{**} = 0(x)$.

Also, for $x, y \in A$, $x \vee 0(y) = x \vee y^{**} = x^* \rightarrow y^{**} = (\text{by } (c_{11})) y^* \rightarrow x^{**} = y^* \rightarrow 0(x) = y \vee 0(x)$.

3. For $a \in A$ and $D \in S(A)$, the map $f_a : D \rightarrow A$ defined by $f_a(x) = x \vee a$ for every $x \in D$ is a multiplier on A (called *principal*).

Indeed, for $b \in A$, $x, y \in D$, we have $f_a(b \vee x) = (b \vee x) \vee a = b \vee (x \vee a)$ (by (c_{16})) $= b \vee f_a(x)$, $f_a(x^{**}) = (x^{**}) \vee a = x^{****} \rightarrow a = x^* \rightarrow a = f_a(x)$ and $x \vee f_a(y) = x \vee (y \vee a) = y \vee (x \vee a)$ (by (c_{16})) $= y \vee f_a(x)$.

Obs. If $\text{dom}(f_a) = A$, we denote f_a by $\bar{f}_a$.

LEMMA 3.4. If $f : D \rightarrow A$ is a multiplier on A ($D \in S(A)$), then:

$$(i) \quad f(1) = 1$$

$$(ii) \quad \text{For every } x \in D, x^{**} \leq f(x).$$

PROOF. (i). If in (a_4) we put $a = 1$, we obtain that for every $x \in D$, $f(1 \vee x) = 1 \vee f(x) \Leftrightarrow f(1) = 1$.

(ii). If in (a_4) we put $a = x$, we obtain that for every $x \in D$, $f(x \vee x) = x \vee f(x) \Leftrightarrow f(x^{**}) = x^* \rightarrow f(x) \Leftrightarrow f(x) = x^* \rightarrow f(x)$ (by (a_5)) $\Rightarrow x \leq f(x) \Rightarrow x^* \rightarrow x \leq x^* \rightarrow f(x)$ (by (c_9)) $\Rightarrow x^{**} \leq f(x)$ (by (c_{12}) and (a_4)).

For $D \in S(A)$, we denote $M(D, A) = \{f : D \mapsto A / f \text{ is a multiplier on } A\}$ and $M(A) = \bigcup_{D \in S(A)} M(D, A)$.

If $D_1, D_2 \in S(A)$ and $f_i \in M(D_i, A)$, $i = 1, 2$, we define $f_1 \rightarrow f_2 : D_1 \cap D_2 \mapsto A$ by $(f_1 \rightarrow f_2)(x) = f_1(x) \rightarrow f_2(x)$ for every $x \in D_1 \cap D_2$ (clearly $D_1 \cap D_2 \in S(A)$).

LEMMA 3.5. $f_1 \rightarrow f_2 \in M(D_1 \cap D_2, A)$.

PROOF. If $a \in A$ and $x, y \in D_1 \cap D_2$, then $(f_1 \rightarrow f_2)(a \vee x) = f_1(a \vee x) \rightarrow f_2(a \vee x) = (a \vee f_1(x)) \rightarrow (a \vee f_2(x)) = (\text{by } (c_{15})) a \vee (f_1(x) \rightarrow f_2(x)) = a \vee (f_1 \rightarrow f_2)(x)$, $(f_1 \rightarrow f_2)(x^{**}) = f_1(x^{**}) \rightarrow f_2(x^{**}) = f_1(x) \rightarrow f_2(x) = (f_1 \rightarrow f_2)(x)$ and $x \vee (f_1 \rightarrow f_2)(y) = x \vee (f_1(y) \rightarrow f_2(y)) = (x \vee f_1(y)) \rightarrow (x \vee f_2(y)) = (y \vee f_1(x)) \rightarrow (y \vee f_2(x)) = y \vee (f_1(x) \rightarrow f_2(x)) = y \vee (f_1 \rightarrow f_2)(x)$.

COROLLARY 3.6. $(M(A), \rightarrow, 1, 0)$ is a bounded Hilbert algebra.

PROOF. The fact that $M(A)$ is a Hilbert algebra follows immediately from lemma 3.5.

If $D \in S(A)$, $f \in M(D, A)$ and $x \in D$, then $0(x) = x^{**} \leq f(x) \leq 1 = 1(x)$ (by lemma 3.4.).

LEMMA 3.7. The map $v_A : A \mapsto M(A)$, defined by $v_A(a) = \bar{f}_a$ for every $a \in A$ is a morphism of Hilbert algebras.

PROOF. If $a, b, x \in A$, we have $(\bar{f}_a \rightarrow \bar{f}_b)(x) = \bar{f}_a(x) \rightarrow \bar{f}_b(x) = (x \vee a) \rightarrow (x \vee b) = (\text{by } (c_{15})) x \vee (a \rightarrow b) = \bar{f}_{a \rightarrow b}(x)$, so $v_A(a) \rightarrow v_A(b) = v_A(a \rightarrow b)$.

Also, $v_A(0) = 0$ since $\bar{f}_0(x) = x \vee 0 = x^{**} = 0(x)$ for every $x \in A$.

DEFINITION 3.8. $D \subseteq A$ is called *regular* if for every $x, y \in A$ such that $t \vee x = t \vee y$ for every $t \in D$, then $x = y$.

For example, A is a regular subset of A since if $x, y \in A$ and $t \vee x = t \vee y$ for every $t \in A$, then for $t = 0$ we obtain $0 \vee x = 0 \vee y \Leftrightarrow 1 \rightarrow x = 1 \rightarrow y \Leftrightarrow x = y$ (more generally, every subset of A which contains 0 is regular).

We denote $R(A) = \{D \subseteq A / D \text{ is a regular subset of } A\}$.

LEMMA 3.9. If $D_1, D_2 \in S(A) \cap R(A)$, then $D_1 \cap D_2 \in S(A) \cap R(A)$.

PROOF. Let $x, y \in A$ such that $t \vee x = t \vee y$ for every $t \in D_1 \cap D_2$.

By (c_{16}) we deduce that for every $t_1, t_2 \in A$ we have $(t_1 \vee t_2) \vee 0 = (t_2 \vee t_1)$

$\mathbb{W} \mathbf{0} = t_1 \mathbb{W} (t_2 \mathbb{W} \mathbf{0})$ and so, if $t_i \in D_i$, $i = 1, 2$, are arbitrary, then $t = (t_1 \mathbb{W} t_2) \mathbb{W} \mathbf{0} = (t_2 \mathbb{W} t_1) \mathbb{W} \mathbf{0} \in D_1 \cap D_2$ (since $t_1 \mathbb{W} t_2 \in D_2$ and by lemma 3.2, $t = (t_1 \mathbb{W} t_2) \mathbb{W} \mathbf{0} = (t_1 \mathbb{W} t_2)^{**} \in D_2$).

Since $t \mathbb{W} x = t \mathbb{W} y$, we obtain $((t_1 \mathbb{W} t_2) \mathbb{W} \mathbf{0}) \mathbb{W} x = ((t_1 \mathbb{W} t_2) \mathbb{W} \mathbf{0}) \mathbb{W} y \Leftrightarrow t_1 \mathbb{W} ((t_2 \mathbb{W} \mathbf{0}) \mathbb{W} x) = t_1 \mathbb{W} ((t_2 \mathbb{W} \mathbf{0}) \mathbb{W} y)$. Since $t_1 \in D_1$ is arbitrary and $D_1 \in S(A) \cap R(A)$, we obtain that $(t_2 \mathbb{W} \mathbf{0}) \mathbb{W} x = (t_2 \mathbb{W} \mathbf{0}) \mathbb{W} y \Leftrightarrow t_2 \mathbb{W} (\mathbf{0} \mathbb{W} x) = t_2 \mathbb{W} (\mathbf{0} \mathbb{W} y) \Leftrightarrow t_2 \mathbb{W} x = t_2 \mathbb{W} y$ (since $\mathbf{0} \mathbb{W} x = 1 \rightarrow x = x$ and $\mathbf{0} \mathbb{W} y = 1 \rightarrow y = y$). Since $t_2 \in D_2$ is arbitrary and $D_2 \in S(A) \cap R(A)$ we obtain that $x = y$, hence $D_1 \cap D_2 \in S(A) \cap R(A)$.

REMARK 3.10. By lemma 3.9., we deduce that $M_r(A) = \{f \in M(A) / \text{dom}(f) \in S(A) \cap R(A)\}$ is a Hilbert subalgebra of $M(A)$.

DEFINITION 3.11. Given two multipliers f_1, f_2 on A , we say that f_1 extends f_2 if $\text{dom}(f_2) \subseteq \text{dom}(f_1)$ and $f_1(x) = f_2(x)$ for all $x \in \text{dom}(f_2)$; we write $f_2 \leq f_1$ if f_1 extend f_2 .

A multiplier f is called *maximal* if f can not be extend to a strictly larger domain.

LEMMA 3.12. (i) If $f_1, f_2 \in M(A)$, $f \in M_r(A)$ and $f \leq f_1$, $f \leq f_2$, then f_1 and f_2 agree on the $\text{dom}(f_1) \cap \text{dom}(f_2)$.

(ii) Every multiplier $f \in M_r(A)$ can be extend to a maximal multiplier. More precisely, each principal multiplier f_a with $\text{dom}(f_a) \in S(A) \cap R(A)$ can be uniquely extend to a total multiplier $\bar{f}_a$ and each non-principal multiplier can be extended to a maximal non-principal one.

PROOF. (i). If by contrary there exists $t \in \text{dom}(f_1) \cap \text{dom}(f_2)$ such that $f_1(t) \neq f_2(t)$, since $\text{dom}(f) \in R(A)$, then there exists $t' \in \text{dom}(f)$ such that $t' \mathbb{W} f_1(t) \neq t' \mathbb{W} f_2(t) \Leftrightarrow f_1(t' \mathbb{W} t) \neq f_2(t' \mathbb{W} t) \Leftrightarrow f_1((t' \mathbb{W} t)^{**}) \neq f_2((t' \mathbb{W} t)^{**})$ which is contradictory since $t_0 = (t' \mathbb{W} t)^{**} = ((t')^* \rightarrow t)^{**} = (t')^* \rightarrow t^{**} = t^* \rightarrow (t')^{**} = t \mathbb{W} (t')^{**} \in \text{dom}(f)$.

(ii). We first prove that f_a can not be extended to a non-principal multiplier.

Let $D = \text{dom}(f_a) \in S(A) \cap R(A)$, $f_a : D \rightarrow A$ and suppose by contrary that there exists $D' \in S(A)$, $D \subseteq D'$ (hence $D' \in R(A)$) and a non-principal multiplier $f \in M(D', A)$ which extends f_a .

Since f is non-principal, there exists $x_0 \in D'$, $x_0 \notin D$ such that $f(x_0) \neq x_0 \mathbb{W} a$.

Since $D \in R(A)$, then there exists $t \in D$ such that $t \mathbb{W} f(x_0) \neq t \mathbb{W} (x_0 \mathbb{W} a) \Leftrightarrow f(t \mathbb{W} x_0) \neq (t \mathbb{W} x_0) \mathbb{W} a \Leftrightarrow f((t \mathbb{W} x_0)^{**}) \neq (t \mathbb{W} x_0) \mathbb{W} a = ((t \mathbb{W} x_0)^{**}) \mathbb{W} a$. Denoting $t_0 = (t \mathbb{W} x_0)^{**} = (t^* \rightarrow x_0)^{**} = t^{***} \rightarrow x_0^{**} = t^* \rightarrow x_0^{**} = x_0^* \rightarrow t^{**} = x_0 \mathbb{W} t^{**} \in D$ (since $t^{**} \in D$).

We obtain that $f(t_0) \neq t_0 \mathbb{W} a$ which is contradictory since $f_a \leq f$. Hence f_a

is uniquely extended by $\bar{f}_a$.

Now, let $f \in M_r(A)$ be non-principal and $M_f = \{(D, g) / D \in S(A), g \in M(D, A), \text{dom}(f) \subseteq D \text{ and } g|_{\text{dom}(f)} = f\}$ (clearly, if $(D, g) \in M_f$, then $D \in S(A) \cap R(A)$).

The set M_f is ordered by $(D_1, g_1) \leq (D_2, g_2)$ iff $D_1 \subseteq D_2$ and $g_2|_{D_1} = g_1$.

Let $\{(D_i, g_i) / i \in I\}$ be a chan in M_f . Then $D' = \bigcup_{i \in I} D_i \in S(A)$ and

$\text{dom}(f) \subseteq D'$. So, $g' : D' \rightarrow A$ defined by $g'(x) = g_i(x)$ if $x \in D_i$ is correctly defined (since if $x \in D_i \cap D_j$ then by (i), $g_i(x) = g_j(x)$).

Clearly $g' \in M(D', A)$ and $g'|_{\text{dom}(f)} = f$ (since if $x \in \text{dom}(f) \subseteq D'$, then $x \in D'$ and so there exists $i \in I$ such that $x \in D_i$, hence $g'(x) = g_i(x) = f(x)$).

So, (D', g') is an upper bound for the family $\{(D_i, g_i) / i \in I\}$, hence by Zorn's lemma, M_f contains at least one maximal multiplier h which extends f .

Since f is non-principal and h extends f , h is also non-principal.

On the Hilbert algebra $M_r(A)$ we consider the relation ρ_A defined by $(f_1, f_2) \in \rho_A$ iff f_1 and f_2 agree on the intersection of their domains.

LEMMA 3.13. ρ_A is a congruence on $M_r(A)$.

PROOF. The reflexivity and the symmetry of ρ_A are immediately; to prove the transitivity of ρ_A , let $(f_1, f_2), (f_2, f_3) \in \rho_A$. Therefore f_1, f_2 and respectively f_2, f_3 agree on the intersection of their domains. If by contrary, there exists $x_0 \in \text{dom}(f_1) \cap \text{dom}(f_3)$ such that $f_1(x_0) \neq f_3(x_0)$, since $\text{dom}(f_2) \in R(A)$, there exists $t \in \text{dom}(f_2)$ such that $t \vee f_1(x_0) \neq t \vee f_3(x_0) \Leftrightarrow f_1(t \vee x_0) \neq f_3(t \vee x_0) \Leftrightarrow f_1((t \vee x_0)^{**}) \neq f_3((t \vee x_0)^{**})$ which is contradictory since $(t \vee x_0)^{**} \in \text{dom}(f_1) \cap \text{dom}(f_2) \cap \text{dom}(f_3)$ (see the proof of lemma 3.9.).

The compatibility of ρ_A with $\rightarrow$ is immediately.

For $f \in M_r(A)$ we denote by $[f]$ this congruence class and $A'' = M_r(A)/\rho_A$.

LEMMA 3.14. The map $\bar{v}_A : A \rightarrow A''$ defined by $\bar{v}_A(a) = [f_a]$ is an injective morphism of Hilbert algebras and $\bar{v}_A(A) \in R(A'')$.

PROOF. The fact that $\bar{v}_A$ is a morphism of Hilbert algebras follows from lemma 3.7.

To prove the injectivity of $\bar{v}_A$, let $a, b \in A$ such that $\bar{v}_A(a) = \bar{v}_A(b)$. Then $[f_a] = [f_b] \Leftrightarrow (f_a, f_b) \in \rho_A \Leftrightarrow f_a(x) = f_b(x)$ for every $x \in A \Leftrightarrow x \vee a = x \vee b$ for every $x \in A \Leftrightarrow a = b$.

To prove $\bar{v}_A(A) \in R(A'')$, if by contrary there exist $f_1, f_2 \in M_r(A)$ such that $[f_1] \neq [f_2]$ (that is there exists $x_0 \in \text{dom}(f_1) \cap \text{dom}(f_2)$ such that $f_1(x_0) \neq f_2(x_0)$) and $[f_a] \vee [f_1] = [f_a] \vee [f_2] \Leftrightarrow [f_a \vee f_1] = [f_a \vee f_2]$ for every a

$\in A$, then in particular for $a = x_0$ we obtain that for every $x \in \text{dom}(f_1) \cap \text{dom}(f_2)$, $(\bar{f}_{x_0} \vee f_1)(x) = (\bar{f}_{x_0} \vee f_2)(x) \Leftrightarrow (\bar{f}_{x_0}^* \rightarrow f_1)(x) = (\bar{f}_{x_0}^* \rightarrow f_2)(x) \Leftrightarrow ((\bar{f}_{x_0}(x) \rightarrow 0(x)) \rightarrow f_1(x)) = ((\bar{f}_{x_0}(x) \rightarrow 0(x)) \rightarrow f_2(x)) \Leftrightarrow ((x^* \rightarrow x_0) \rightarrow x^{**}) \rightarrow f_1(x) = ((x^* \rightarrow x_0) \rightarrow x^{**}) \rightarrow f_2(x) \Leftrightarrow (x^* \rightarrow x_0^*) \rightarrow f_1(x) = (x^* \rightarrow x_0^*) \rightarrow f_2(x)$; in particular for $x = x_0$ we obtain $1 \rightarrow f_1(x_0) = 1 \rightarrow f_2(x_0) \Leftrightarrow f_1(x_0) = f_2(x_0)$ which is contradictory.

REMARK 3.15. Since for every $a \in A$, $\bar{f}_a$ is the unique maximal multiplier on $[\bar{f}_a]$ (by lemma 3.12.), we can identify $[\bar{f}_a]$ with f_a .

So, since $\bar{v}_A$ is an injective morphism of Hilbert algebras, the elements of A can be identified with the elements of the set $\{\bar{f}_a / a \in A\}$.

LEMMA 3.16. In view of the identifications made above, if $[f] \in A''$ (with $f \in M_r(A)$ and $D = \text{dom}(f) \in S(A) \cap R(A)$), then:

$$D \subseteq \{a \in A / \bar{f}_a \vee [f] \in A\}.$$

PROOF. Let $a \in D$. If by contrary $\bar{f}_a \vee [f] \notin A$ (that is $[\bar{f}_a \vee f] \notin \bar{v}_A(A)$), then $\bar{f}_a \vee f$ is a non-principal multiplier on A .

Then, by lemma 3.12., (ii), $\bar{f}_a \vee f$ can be extended to a non-principal maximal multiplier $\bar{f}: \bar{D} \rightarrow A$, with $\bar{D} \in S(A)$.

Thus, $D \subseteq \bar{D}$ and for every $x \in D$, $\bar{f}(x) = (\bar{f}_a \vee f)(x) = (\bar{f}_a^* \rightarrow f)(x) = ((\bar{f}_a \rightarrow 0) \rightarrow f)(x) = (\bar{f}_a(x) \rightarrow 0(x)) \rightarrow f(x) = ((x^* \rightarrow a) \rightarrow (x^* \rightarrow 0)) \rightarrow f(x) = (x^* \rightarrow a^*) \rightarrow f(x)$.

Thus, for every $x \in D$, $x^* \rightarrow \bar{f}(x) = x^* \rightarrow ((x^* \rightarrow a^*) \rightarrow f(x)) \Leftrightarrow \bar{f}(x^* \rightarrow x) = (x^* \rightarrow a^*) \rightarrow (x^* \rightarrow f(x)) \Leftrightarrow \bar{f}(x^{**}) = x^* \rightarrow (a^* \rightarrow f(x)) \Leftrightarrow \bar{f}(x) = a^* \rightarrow (x^* \rightarrow f(x)) = a^* \rightarrow f(x) = a \vee f(x)$.

Since $a \in D$, then by (a₆) we deduce that for every $x \in D$, $\bar{f}(x) = a \vee f(x) = x \vee f(a)$, that is $\bar{f}|_D$ is principal which is contradictory with the assumption that $\bar{f}$ is non-principal.

4. Maximal Hilbert algebra of quotients

DEFINITION 4.1. A Hilbert algebra A' is called *Hilbert algebra of fractions* of A if

- (a₇) A is a Hilbert subalgebra of A'
- (a₈) For every $a', b', c' \in A'$, $a' \neq b'$, there exists $a \in A$ such that $a \vee a' \neq a \vee b'$ and $a \vee c' \in A$.

So, every Hilbert algebra is a Hilbert algebra of fractions of itself.

As a notational convenience, we write $A \leq A'$ to indicate that A' is a Hilbert algebra of fractions for A .

DEFINITION 4.2. M is the *maximal Hilbert algebra of quotients* of A if $A \leq M$ and for every A' with $A \leq A'$, there exists an injective morphism of Hilbert algebras $i: A' \rightarrow M$.

LEMMA 4.3. Let $A \leq A'$; then for every $a', b' \in A'$, $a' \neq b'$, and any finite sequence $c'_1, c'_2, \dots, c'_n \in A'$, there exists $a \in A$ such that $a \vee a' \neq a \vee b'$ and $a \vee c'_i \in A$ for $i = 1, 2, \dots, n$.

PROOF. Assume lemma holds true for $n-1$. So we may find $b \in A$ such that $b \vee a' \neq b \vee b'$ and $b \vee c'_i \in A$ for $i = 1, 2, \dots, n-1$. Since $A \leq A'$, we find $c \in A$ such that $c \vee (b \vee a') \neq c \vee (b \vee b')$ and $c \vee c'_n \in A$. The element $a = b \vee c \in A$ has the required properties.

LEMMA 4.4. Let $A \leq A'$ and $a' \in A'$. Then $D_{a'} = \{a \in A / a \vee a' \in A\} \in S(A) \cap R(A)$.

PROOF. If $a \in A$ and $x \in D_{a'}$, then $x \vee a' \in A$ and since $(a \vee x) \vee a' = a \vee (x \vee a') \in A$ it follows $a \vee x \in D_{a'}$, hence $D_{a'} \in S(A)$. To prove $D_{a'} \in R(A)$, let $x, y \in A$ such that $a \vee x = a \vee y$ for every $a \in D_{a'}$.

If by contrary $x \neq y$, since $A \leq A'$, there exists $a_0 \in A$ such that $a_0 \vee a' \in A$ (that is $a_0 \in D_{a'}$) and $a_0 \vee x \neq a_0 \vee y$, which is contradictory.

THEOREM 4.5. A'' (defined in section 3) is the maximal Hilbert algebra of quotients of A .

PROOF. The fact that A is a Hilbert subalgebra of A'' follows from lemma 3.14. and remark 3.15.

To prove $A \leq A''$, let $[f], [g], [h] \in A''$ (with $f, g, h \in M_r(A)$) such that $[g] \neq [h]$ (that is, there exists $x_0 \in \text{dom}(g) \cap \text{dom}(h)$ such that $g(x_0) \neq h(x_0)$).

Put $D = \text{dom}(f) \in S(A) \cap R(A)$ and $D_{[f]} = \{a \in A / \bar{f}_a \vee [f] \in A\}$. Then by lemma 3.16., $D \subseteq D_{[f]}$.

If suppose that for every $a \in D$, $\bar{f}_a \vee [g] = \bar{f}_a \vee [h]$, then $[\bar{f}_a \vee g] = [\bar{f}_a \vee h]$, hence for every $x \in \text{dom}(g) \cap \text{dom}(h)$ we have $(\bar{f}_a \vee g)(x) = (\bar{f}_a \vee h)(x) \Leftrightarrow$ (analogously than as in the proof of lemma 3.14.) $\Leftrightarrow$

$$(x^* \rightarrow a^*) \rightarrow g(x) = (x^* \rightarrow a^*) \rightarrow h(x).$$

From this last relation we deduce that for every $x \in \text{dom}(g) \cap \text{dom}(h)$

$$\begin{aligned} x^* \rightarrow ((x^* \rightarrow a^*) \rightarrow g(x)) &= x^* \rightarrow ((x^* \rightarrow a^*) \rightarrow h(x)) \Rightarrow (x^* \rightarrow a^*) \\ \rightarrow (x^* \rightarrow g(x)) &= (x^* \rightarrow a^*) \rightarrow (x^* \rightarrow h(x)) \Rightarrow x^* \rightarrow (a^* \rightarrow g(x)) = x^* \rightarrow (a^* \rightarrow h(x)) \\ \Rightarrow a^* \rightarrow (x^* \rightarrow g(x)) &= a^* \rightarrow (x^* \rightarrow h(x)) \Rightarrow a^* \rightarrow g(x) = a^* \rightarrow h(x) \Leftrightarrow a \vee g(x) \\ &= a \vee h(x). \end{aligned}$$

Since $D \in R(A)$ we deduce that $g(x) = h(x)$ for every $x \in \text{dom}(g) \cap \text{dom}(h) \Leftrightarrow [g] = [h]$ which is contradictory.

Hence, if $[g] \neq [h]$, then there exists $a \in D$ such that $\bar{f}_a \vee [g] \neq \bar{f}_a \vee [h]$.

But for this $a \in D$ we have clearly $\bar{f}_a \vee [f] \in A$ (since $D \subseteq D_{[f]}$), hence $A \leq A''$.

To prove the „maximality“ of A'' , let A' be a Hilbert algebra such that $A \leq A'$.

Then A' is embeded in A'' by virtue of $i: A' \rightarrow A''$ defined by $i(a') = [f_{a'}]$ for every $a' \in A'$, where $\text{dom}(f_{a'}) = D_{a'}$ (see lemma 4.4.).

Clearly $f_{a'} \in M_r(A)$ and i is a morphism of Hilbert algebras (see lemma 3.7.).

To prove the injectivity of i , let $a', b' \in A'$ such that $[f_{a'}] = [f_{b'}] \Leftrightarrow f_{a'}(x) = f_{b'}(x)$ for every $x \in D_{a'} \cap D_{b'}$. If $a' \neq b'$, by lemma 4.3, (since $A \leq A'$) there exists $a \in A$ such that $a \vee a', a \vee b' \in A$ and $a \vee a' \neq a \vee b'$ which is contradictory (since $a \vee a', a \vee b' \in A \Rightarrow a \in D_{a'} \cap D_{b'}$).

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