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SIGNATURE OF TORUS LINKS

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Denote by $L(p, q)$ the (p, q) torus link, oriented so that each component represents the same homology class of the torus. The computation (or estimation) of $\sigma(p, q)$ has been attempted by various people with various motivation (see e.g. [3, 5, 8]). A more general result was obtained by Gordon, Litherland and Murasugi [2], where they proved several recursive formulae for $\sigma(p, q)$. Their result is equivalent to the following three theorems, which, together with some well-known facts ($\sigma(p, q) = \sigma(q, p) = \sigma(p, -q)$, $\sigma(p, 1) = \sigma(p, 0) = 0$), will allow one to compute $\sigma(p, q)$ for all p, q .

THEOREM 1. *If both p, q are even and positive,*

$$\sigma(p, q) = \frac{1}{2}pq - 1.$$

THEOREM 2. *If p is odd and positive,*

$$\sigma(p, 2kp+q) = k(p^2 - 1) + \sigma(p, q).$$

THEOREM 3. *If p is even, and $0 < q < p$, then*

$$(a) \quad \sigma(p, q) = \frac{1}{2}p^2 - 2 + \sigma(p, q-p),$$

$$(b) \quad \sigma(p, kp+q) = \frac{1}{2}kp^2 + \sigma(p, q), \quad k > 0.$$

REMARK 1. Theorem 1 is a new result in the sense that it gives an explicit formula for the signature of torus links of even components. Theorem 2 is a combination of two formulae in [2, Thm 5.2]. Note that q and k can be any integers. Theorem 3 is also an improvement of [2, Thm 5.2] (where formula (b) was proved only for even k). It will sometimes allow quicker computation. For example, from Theorem 3 we have $\sigma(p, kp \pm 1) = \frac{1}{2}kp^2 - 1 \pm 1$, ($k, p > 0, p$ even). This is not so obvious from [2] when k is odd.

In this paper we will give a new proof of the above theorems. Another result is

Theorem 4 below, which gives a relation between the signature of some links. Theorem 2 is a direct consequence of Theorem 4.

Our proof of the theorems is based on Gordon and Litherland's algorithm [1]. We describe it first.

1. The algorithm of Gordon and Litherland

Let V be a surface in S^3 , not necessarily orientable with ∂V a link L . We assume V has no closed components. If $a \in H_1(V)$, then $2a$ can be pushed off V , obtaining τa . (If a preserves orientation, we push a copy of a to each side of V .)

Let $\{a_1, \dots, a_n\}$ be a basis of $H_1(V)$. Define a matrix related to this basis as $G = (lk(a_i, \tau a_j))_{n \times n}$. Denote by $\text{sign } G$ the signature of G . (G was proved to be symmetric.)

The link $L = \partial V = l_1 \cup \dots \cup l_r$ can be pushed off V to obtain a link $L^V = l_1^V \cup \dots \cup l_r^V$. Suppose L is oriented.

Define

$$\bar{e}(V) = -lk(L, L^V) = -\sum_{i,j=1}^r lk(l_i, l_j^V).$$

COROLLARY 5". ([1]). *With the above notation,*

$$\sigma(L) = \text{sign } G + \frac{1}{2} \bar{e}(V).$$

We also need the following result, which follows directly from Theorem 12 of [1].

PROPOSITION 1. *If L is a (p, q) cable link of a knot K with p even, then $\sigma(L) = \sigma(p, q)$.*

Suppose the g.c.d. $(p, q) = k$ is even. Then $p = kr$, $q = ks$ for some integer r, s , $(r, s) = 1$. In this case $L(p, q)$ is a (k, krs) cable of $L(r, s)$. Hence we have

COROLLARY 2. *With the above assumption,*

$$\sigma(p, q) = \sigma(k, krs).$$

2. Proof of Theorem 1

We rewrite Theorem 1 as

THEOREM 1'. *If both p, q are positive,*

$$\sigma(2p, 2q) = 2pq - 1.$$

PROOF. First we suppose that $q = pr$, r an integer. In this case, $L(2p, 2q)$ has a spanning surface V consisting of p bands lying on the torus. (See Figure 1, where $p = q = 2$).

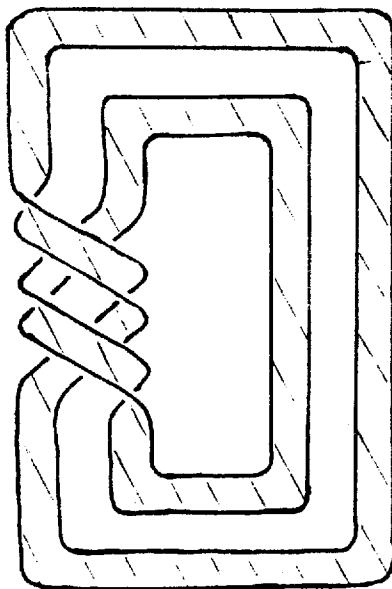


Figure 1

Let a_i be represented by a boundary curve of the i -th band. Then

$$\text{lk}(a_i, \tau a_j) = -2r.$$

Hence $\text{sign } G = -1$.

From the Figure, one can easily verify that $\text{lk}(l_i, l_j^V) = -r$. Thus

$$\bar{e}(V) = - \sum_{i,j=1}^{2p} \text{lk}(l_i, l_j^V) = (2p)^2 r.$$

Therefore

$$\sigma(2p, 2q) = \text{sign } G + \frac{1}{2} \bar{e}(V) = 2p^2 r - 1 = 2pq - 1.$$

Now suppose $(p, q) = k < p$, $p = kr$, $q = ks$. By Corollary 2,

$$\begin{aligned} \sigma(2p, 2q) &= \sigma(2kr, 2ks) = \sigma(2k, 2krs) = 2k^2 rs - 1 \\ &= 2pq - 1. \end{aligned}$$

This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let T be a solid torus in S^3 . Cutting T along a meridian disc, twisting it k times left-hand-wise, then gluing back, we obtain a map $f: T \rightarrow T$, called a k -twist map of T . Figure 2 illustrates a 1-twist map of the trivial solid torus.

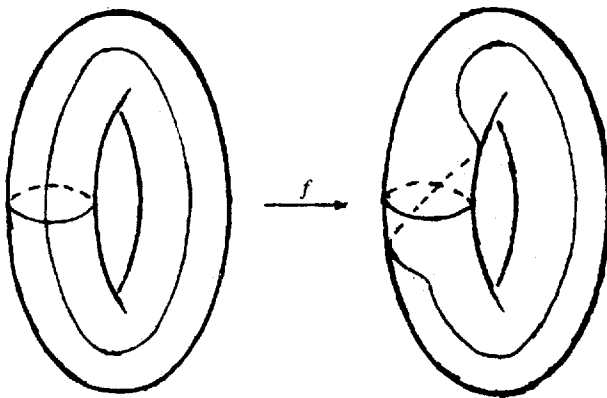


Figure 2

Suppose L, L' are two oriented links in the interior of T . Using the method in ([7], P. 266), one can verify the following Lemma. (We omit the proof.)

LEMMA 1. *If f is a k -twist map, then*

$$\text{lk}(f(L), f(L')) = \text{lk}(L, L') - k \cdot \text{lk}(L, m) \cdot \text{lk}(L', m),$$

where m is a meridian on ∂T .

Put $L(p, q)$ in a solid torus T in the natural way. Let f be a k -twist map of T . Then $f(L(p, q)) = L(p, kp+q)$. Thus Theorem 2 is a special case of the following general result.

THEOREM 4. *Let L be an oriented link lying in a solid torus T . Suppose $p = \text{lk}(L, m)$ is odd (where m is a meridian of ∂T). Then*

$$\sigma(f(L)) = \sigma(L) + k(p^2 - 1),$$

where f is a $2k$ -twist map.

PROOF. We need only to prove the Theorem for $k = 1$.

Let D be a meridian disc of T with $\partial D = m$. Let $T_1 = D \times [-1, 1]$ be a

neighborhood of D in T . We may assume that $T_1 \cap L$ consists of $2n+1$ ($\geq p$) parallel lines. Construct a connected spanning surface V of L such that near T_1 the surface V looks like that drawn in Figure 3 (where $n=2$).

Pushing m slightly into T , we obtain a curve m' such that m' is disjoint from V and surrounds the n bands. Choose a basis $\{a_1, \dots, a_r\}$ of $H_1(V)$ with $\text{lk}(a_i, m') = \delta_{1i}$. Suppose the matrix related to this basis is $G = (g_{ij})$.

Choose a 2-twist map f so that $f|_{T-T_1} = \text{id}$. We construct a surface V_1 spanning $f(L)$ such that V_1 is as drawn in Figure 4 in a neighborhood of T_1 , and is equal to V out of the neighborhood.

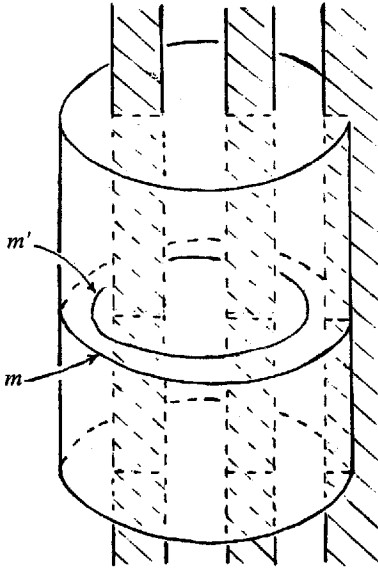


Figure 3

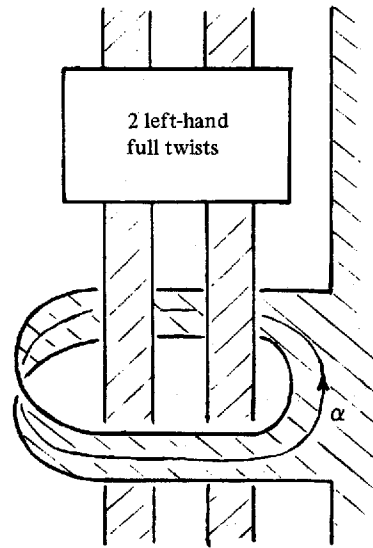


Figure 4

We choose a basis $\{a'_1, \dots, a'_{r+1}\}$ for $H_1(V_1)$ as follows: Let $\{a_1, \dots, a_r\}$ be the basis for $H_1(V)$ chosen as above. Suppose a_i is represented by a curve α_i . Then a'_i is represented by the curve $f(\alpha_i)$, $i \leq r$, and a'_{r+1} by the curve α drawn in Figure 4. Let G' be the matrix related to this basis. By Lemma 1 and direct computation, we have

$$G' = \begin{pmatrix} g_{11}-4 & g_{12} & \cdots & g_{1r} & \pm 2 \\ g_{21} & g_{22} & \cdots & g_{2r} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ g_{r1} & g_{r2} & \cdots & g_{rr} & 0 \\ \pm 2 & 0 & \cdots & 0 & -1 \end{pmatrix}$$

Clearly, G' is congruent to $G \oplus (-1)$. Hence $\text{sign } G' = \text{sign } G - 1$. Using Lemma 1, one can show that

$$\text{lk}(f(L), f(L)^V) = \text{lk}(L, L^V) - 2p^2.$$

Therefore,

$$\sigma(f(L)) = \text{sign } G' - \frac{1}{2} \text{lk}(f(L), f(L)^V) = \sigma(L) + p^2 - 1.$$

4. Proof of Theorem 3

The following Proposition can also be proved by the algorithm of Gordon and Litherland. We omit the proof because it is a special case of [4, Thm 2].

PROPOSITION 2. *Let L be an oriented link lying in a solid torus T such that $p = \text{lk}(L, m)$ is even (where m is a meridian of ∂T). Let f be a k -twist map, $k > 0$. Then*

$$\frac{1}{2}kp^2 - 2 \leq \sigma(f(L)) - \sigma(L) \leq \frac{1}{2}kp^2.$$

We prove Theorem 3 by induction. If q is even, the result follows from Theorem 1. So we assume q is odd. We have ([6])

$$\sigma(2, 2k+1) = 2k, \quad k > 0.$$

So the result is true for $p = 2$. Suppose it is true for $p < 2n$.

(a). Without loss of generality, we may assume $0 < q < n$. (Otherwise consider $q' = 2n - q$.) By Theorem 2,

$$\begin{aligned} \sigma(2n, q) + \sigma(2n, 2n - q) &= \sigma(q, 2n) + \sigma(2n - q, 2n) = \\ &= q^2 - 1 + (2n - q)^2 - 1 + \sigma(q, 2n - 2q) + \sigma(2n - q, 2q - 2n). \end{aligned}$$

By inductive assumption,

$$\begin{aligned} -\sigma(2n - q, 2q - 2n) &= \sigma(2n - 2q, 2n - q) = \\ &= \frac{1}{2}(2n - 2q)^2 + \sigma(2n - 2q, q). \end{aligned}$$

(This is obviously true for $q = n$). Thus

$$\begin{aligned} \sigma(2n, q) + \sigma(2n, 2n - q) &= \\ &= q^2 + (2n - q)^2 - 2 - 2(n - q)^2 = 2n^2 - 2. \end{aligned}$$

(b). By Proposition 2 and (a),

$$\begin{aligned}
\sigma(2n, 2nk+q) &\geq \sigma(2n, q-2n) + 2(k+1)n^2 - 2 \\
&= \sigma(2n, q) - 2n^2 + 2 + 2(k+1)n^2 - 2 \\
&= \sigma(2n, q) + 2kn^2.
\end{aligned}$$

On the other hand, (again by Proposition 2),

$$\sigma(2n, 2nk+q) \leq \sigma(2n, q) + 2kn^2.$$

Therefore the equality holds. Q.E.D.

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