A note on the ascending number

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A NOTE ON THE ASCENDING NUMBER

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Abstract

For an oriented diagram of a knot, the minimum number of crossing changes required to convert the diagram into a descending diagram is said to be the warping degree. The minimum number of warping degrees of all diagrams of a knot is said to be the ascending number of the knot. In this paper, we study the relation between the ascending number and the warping degree. We determine the ascending number of certain families of knots. We prove that the ascending number equals the half of the crossing number minus one if and only if a knot is the $(2, p)$-torus knot. We examine the determinations of the ascending numbers of prime knots up to ten crossings.

1. Introduction

Throughout this paper, a knot is an oriented closed curve in $\mathbf{S}^3$. We use standard knot diagrams based on Rolfsen's knot table [9] unless otherwise stated. For a knot K and a diagram D of K, we denote $c(D)$ the crossing number of D and $c(K)$ the crossing number of K. A crossing change is a local move of a diagram as illustrated in Figure 1. The *unknotting number* $u(K)$ of a knot K is the minimum number of crossing changes required to convert a knot into the unknot. A knot diagram is *based* if a base point (different from the crossing points) is specified on the diagram.

Ozawa [8] introduces the ascending number of a link. A knot diagram is said to be *descending* if we meet each crossing as an overcrossing first when we go along the diagram with an orientation by starting from a base point on the diagram. Let K be a knot, and D a based oriented diagram of K. The *ascending number* of D is defined as the number of crossing changes required to convert D into a descending diagram, and denoted by $a(D)$. The *ascending number* of K is defined as the minimum number of $a(D)$ over all based oriented diagram D of K, and denoted by $a(K)$.

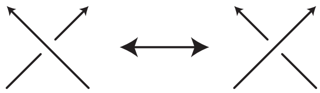


Figure 1.

The useful notion to study the ascending number is the warping degree which is introduced by Kawauchi [4], and lately studied by Shimizu [10]. We study a relation of the ascending number and the warping degree in Section 2. By definition, the ascending number is a generalization of the warping degree. In fact, we have that the ascending number coincides with the warping degree of a knot (Corollary 2.10).

In Section 3, we introduce moves of a diagram in order to determine the ascending number of knots. We determine the ascending number of knots by using such moves.

In Section 4, we study the property of the ascending number of a knot by comparing with the unknotting number. The ascending number is related to the unknotting number. For a diagram D of a knot K, since $a(K)$ times crossing changes convert it into a diagram of the trivial knot, we have $u(K) \leq a(K)$. It is well known that $u(K) \leq \frac{c(K)-1}{2}$, and there is also a result for the ascending number in [8].

Proposition 1.1 ([8]). *Let K be a non-trivial knot. Then $a(K) \leq \frac{c(K)-1}{2}$.*

Furthermore, we have the following.

Theorem 1.2 ([11]). *Let K be a knot. $u(K) = \frac{c(K)-1}{2}$ if and only if K is the $(2,p)$-torus knot for some odd integer $p \neq \pm 1$.*

Theorem 1.3 ([1]). *If K is a knot with $u(K) = \frac{c(K)-2}{2}$, then K is the figure-eight knot, a positive 3-braid knot, a negative 3-braid knot, or a connected sum of the $(2,p)$-torus knot and the $(2,p')$-torus knot for some odd integers $p, p' \neq \pm 1$.*

We give a result similar to Theorem 1.2 for the ascending number.

Theorem 1.4. *Let K be a knot. $a(K) = \frac{c(K)-1}{2}$ if and only if K is the $(2,p)$-torus knot for some odd integer $p \neq 1$.*

Problem 1.5. Characterize knots with $a(K) = \frac{c(K)-2}{2}$.

Generally, it is difficult to characterize the unknotting number one knot. On the other hand, we have the following result for the ascending number.

Theorem 1.6 ([8]). *$a(K) = 1$ if and only if K is a twist knot.*

When a knot K is a twist knot, a torus knot, or a knot in Theorem 1.3, we have $u(K) = a(K)$. On the other hand, we have many knots such that $u(K) \neq a(K)$ by Theorem 1.6, since there are many non-twist knots such that $u(K) = 1$.

PROBLEM 1.7. Characterize knots with $u(K) = a(K)$.

By Ozawa, the ascending number is completely determined for twist knots and torus knots. We study the following problem in Section 5.

PROBLEM 1.8. Determine the ascending number for some family of knots.

The following is an answer of Problem 1.8.

THEOREM 1.9. *Let $C(m)$, $C(m,n)$, and $C(m,n,l)$ be the 2-bridge knots with Conway notation m, mn, and mnl, respectively, where m, n, l are positive integers. Then,*

(1) $a(C(m)) = \frac{1}{2}(m-1)$ *if m is odd;*
(2) $a(C(m,n)) = \frac{1}{2}m$ *if m is even and n is odd;*
(3) $a(C(m,n,l)) = \frac{1}{2}(m+l-1)$ *if m and n are even and l is odd.*

In Section 6, we study the ascending number of a reduced alternating diagram. In Section 7, we give a table of knots up to ten crossings with respect to the ascending number. This paper is based on the master thesis of the second author [5], and also based on the first author's recent paper [2] for Section 7.

2. Ascending number and warping degree

At the beginning, we review the notion of the warping degree according to Shimizu [10]. Let D be an oriented knot diagram. A *base point* b of a knot diagram D is a point on D which is not a crossing point. We denote the pair of D and b by D_b. A crossing point of D_b is said to be a *warping crossing point* if we meet the point as an undercrossing first when we go along D with the orientation by starting from b. The *warping degree* of a based diagram D_b, denoted by $d(D_b)$, is the number of warping crossing points of D_b. The *warping degree* of D, denoted by $d(D)$, is the minimal warping degree for all base points of D. Therefore, $d(D) = \min\{d(D_b) \mid b\}$.

DEFINITION 2.1. The *warping degree* of an oriented knot K is the minimum number of all diagrams of K. Therefore, $d(K) = \min\{d(D) \mid D\}$.

We recall the definition of the ascending number by using the notion of the warping degree.

Definition 2.2. The *ascending number* $a(D)$ of a knot diagram D is the minimum one of $d(D)$ and $d(-D)$. Therefore, $a(D) = \min\{d(D), d(-D)\}$.

Definition 2.3. The *ascending number* $a(K)$ is the minimum one of the ascending number of all diagrams of K. That is, $a(K) = \min\{a(D) \mid D\} = \min\{d(K), d(-K)\}$, where D is a diagram of K.

Our aim in this section is to prove that the ascending number of a knot coincides with the warping degree of the knot. We refer to some results concerning the warping degree.

Lemma 2.4 ([10]). *Let D be a knot diagram.*

(1) *For the base points b and b' of D which are put across an overcrossing as illustrated in* Figure 2 (1), *we have*

$$d(D_{b'}) = d(D_b) + 1.$$

(2) *For the base points b and b' of D which are put across an undercrossing as illustrated in* Figure 2 (2), *we have*

$$d(D_{b'}) = d(D_b) - 1.$$

Lemma 2.5 ([10]). *Let D be an oriented alternating knot diagram. Let b be a base point of D which is just before an overcrossing as illustrated in* Figure 2 (1). *Then,*

$$d(D_b) = d(D).$$

Lemma 2.6 ([10]). *Let D be an oriented knot diagram. For each base point b of D, we have*

$$d(D_b) + d(-D_b) = c(D).$$

Lemma 2.7 ([10]). *Let D be an oriented knot diagram, and $D^\flat$ the diagram obtained from D by crossing changes for all crossings. Then, we have*

$$d(-D) = d(D^\flat).$$

Lemma 2.8. *Let D be a knot diagram, and $D^\#$ the mirror image with respect to the vertical line, see* Figure 3. *Then, we have*

$$d(D) = d(D^\#).$$

Proof. We see that D and $D^\#$ have the same over and under informations as illustrated in Figure 3. □

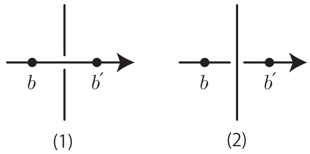


Figure 2.

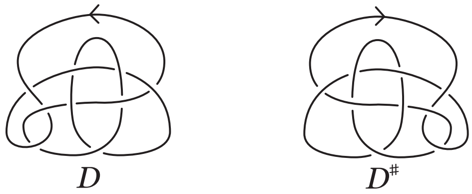


Figure 3.

Theorem 2.9. *Let K be a knot, and $K^\star$ the mirror image of K. Then, we have*

$$d(K) = d(-K) = d(K^\star).$$

Proof. Let D be a diagram of K such that $d(D) = d(K)$, and $D^\flat$ and $D^\#$ the diagrams of $K^\star$ as in Lemmas 2.7 and 2.8. By Lemma 2.8, $d(K) = d(D) = d(D^\#) \geq d(K^\star)$, hence $d(K) \geq d(K^\star)$. Since K is the mirror image of $K^\star$, we also have $d(K) \leq d(K^\star)$. Therefore $d(K) = d(K^\star)$.

Let $-E$ be the diagram of $-K$ such that $d(-E) = d(-K)$. By Lemma 2.7, $d(-K) = d(-E) = d(E^\flat) \geq d(K^\star)$, hence $d(-K) \geq d(K^\star)$. We also have $d(-K) \leq d(K^\star)$ by the same argument as above. Therefore, we have $d(-K) = d(K^\star)$. □

By Theorem 2.9, we have the following immediately.

Corollary 2.10. *For a knot K, we have*

(1) $a(K) = d(K)$;
(2) $a(K) = a(K^\star)$.

3. Moves to decrease the ascending number of a knot diagram

In this section, we introduce a collection of moves that convert a diagram into one with the ascending number as less than before. By these moves and Theorem 1.6, we determine the ascending number of a variety of knots up to ten crossings.

We first consider a tangle containing oriented anti-parallel $2k$ half-twists as illustrated in Figure 4, where k is an integer. We take a base point just before the

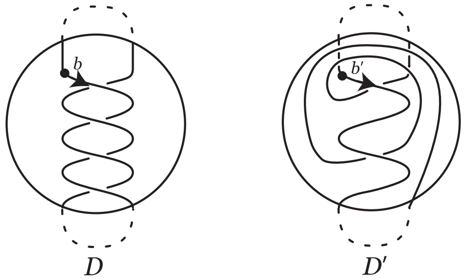


Figure 4.

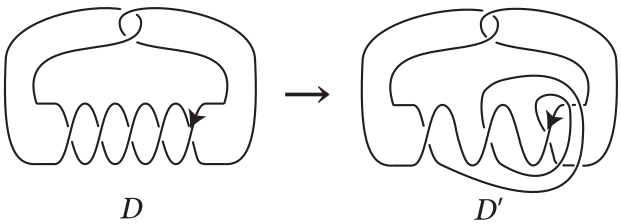


Figure 5.

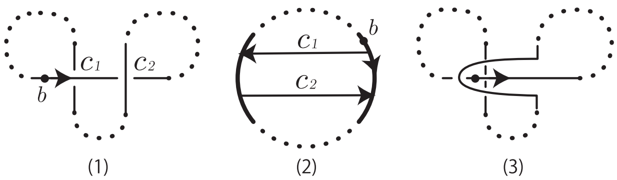


Figure 6.

overcrossing of $2k$ half-twists as in Figure 4. Then D_b has k warping crossings in the portion of the twists. Let $D'_{b'}$ be a based diagram obtained from D_b by a move as illustrated in Figure 4. Then we can remove k warping crossings. D' is said to be obtained by a *whirl move* for D.

We remark that we obtain a diagram of a twist knot such that $a(D) = 1$ by a whirl move. Let D be a diagram of a twist knot as illustrated in Figure 5. Then $a(D) = 4$. Let D' be a diagram obtained from D by a whirl move for D as illustrated in Figure 5. Then $a(D') = 1$.

A chord diagram of D or D_b is a circle with n chords marked on it by line segments, where the preimage of each crossing is connected by a chord. We take the orientation of the chord from the overcrossing to the undercrossing.

We consider two successive alternating crossings. Let D_b be a based diagram as illustrated in Figure 6 (1), and c_1 and c_2 two crossings in D_b. Then, c_2 is a warping crossing of D_b. Then, Figure 6 (2) is a chord diagram corresponding to Figure 6 (1). By a move as Figure 6 (3), we remove the warping crossing c_2.

We generalize such a move as follows. Let D_b be a based oriented diagram, and c a chord of D_b corresponding to warping crossings satisfying the following; c does not intersect chords whose endpoints are on $\overline{bc}$, where $\overline{bc}$ is the subset of the circle of the chord diagram starting from b to one of ∂c with orientation such that $\overline{bc} \cap (b \cup c) = \partial \overline{bc}$. Then we can reduce the warping crossing corresponding to c by pulling over-arc of c into before of b. We call such a portion of a chord diagram a *chord of type* 1. We determine the ascending number for a variety of knots by such moves. In particular, we improve Jablan's result for the following thirty-two knots; 9_{32}, 9_{40}, 10_{50}, 10_{51}, 10_{54}, 10_{61}, 10_{65}, 10_{66}, 10_{84}, 10_{87}, 10_{89}, 10_{90}, 10_{92}, 10_{95}, 10_{98}, 10_{101}, 10_{102}, 10_{104}, 10_{105}, 10_{107}, 10_{110}, 10_{111}, 10_{114}, 10_{120}, 10_{121}, 10_{140}, 10_{144}, 10_{153}, 10_{154}, 10_{155}, 10_{157}, and 10_{158}.

We next consider four successive alternating crossings. Let c_1, c_2, c_3, and c_4 be four successive crossings as in Figure 7. Then, c_2 and c_4 are warping crossings of D when we take a base point just before c_1. Then, we can remove the warping crossings in the five cases as illustrated in Figure 8. We call a portion of a chord diagram as Figure 8 a *chord of type* 2, 3, 4, 5, and 6, respectively. Figure 9 illustrates diagrams corresponding to chords of type 2, 3, 4, 5, and 6, respectively. We can remove the warping crossings c_2 and c_4 by moves as illustrated in Figure 10.

We determine the ascending number for a variety of knots by such moves. In particular, we improve Jablan's result for the following seven knots; 9_{33}, 10_{63}, 10_{67}, 10_{83}, 10_{86}, 10_{88}, and 10_{115}.

Example 3.1. Let K be the knot 9_{33}, and D a diagram of K as illustrated in Figure 11. Then, $a(D) = 4$. Since a chord diagram of D contains a chord of type 6, then K has a diagram D' such that $a(D') = 2$. Since K is not a twist knot, we have $a(K) \geq 2$ by Theorem 1.6. Therefore, $a(K) = 2$.

For the knots 10_{31} and 10_{58}, we can find two chords of type 1.

Example 3.2. Let K be the knot 10_{58}, and D_b a diagram of K as illustrated in Figure 12. Then, $a(D_b) = 4$. Since the chords 2 and 6 are type 1, K has a diagram D' such that $a(D') = 2$. Since K is not a twist knot, hence $a(K) \geq 2$. Therefore, $a(K) = 2$.

We remark that $a(10_{31}) = 2$ by the same argument as Example 3.2.

For the knots 10_{132} and 10_{136}, we cannot find chords of types 1–6 in the diagrams of Rolfsen's knot table. But, we can find a chord of type 1 in the diagrams as illustrated in Figure 13.

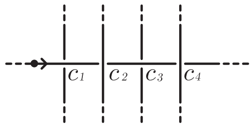


Figure 7.

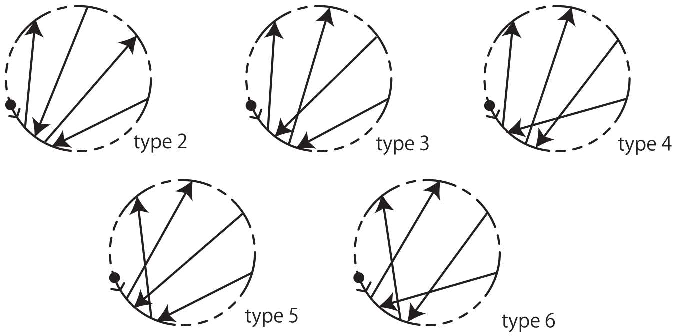


Figure 8.

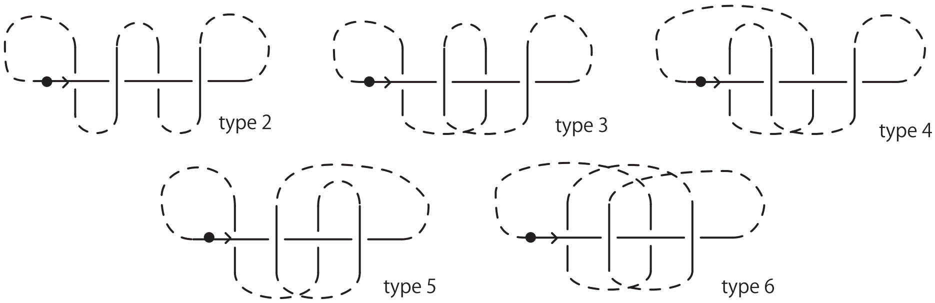


Figure 9.

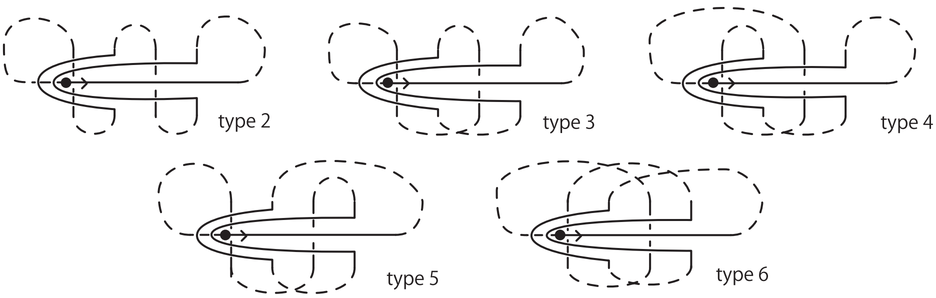


Figure 10.

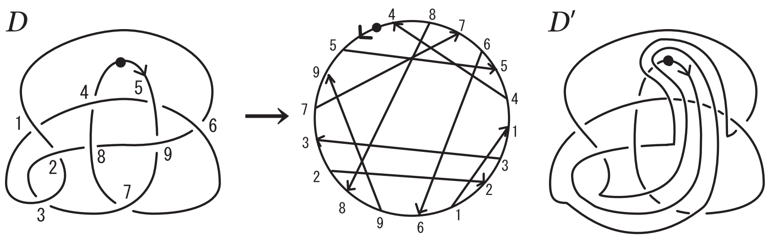


Figure 11.

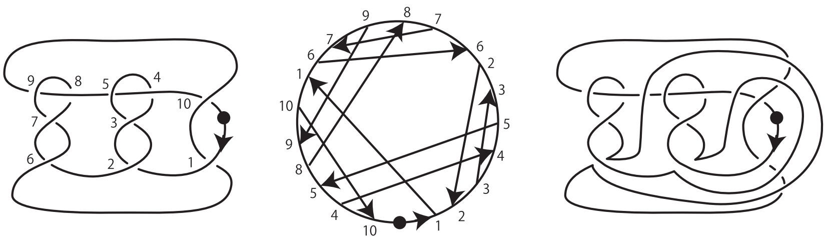
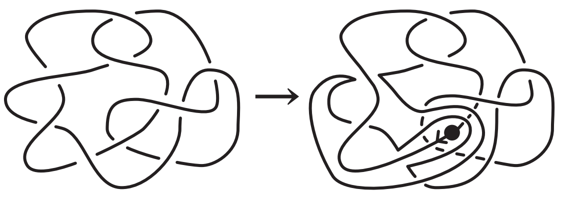


Figure 12.

10_{132}



10_{136}

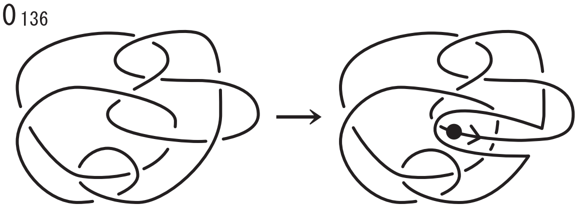


Figure 13.

4. Properties of the ascending number

In this section, we prove Theorem 1.4.

Lemma 4.1. *Let D be a diagram with $c(D) \geq 1$. Then, we have*

$$a(D) \leq \frac{1}{2}(c(D) - 1).$$

Proof. We take a base point b of D just before an overcrossing. By Lemma 2.6, we have $d(D_b) + d(-D_b) = c(D)$. Let b' be the base point of $-D$ as illustrated in Figure 14. By Lemma 2.4, we have $d(-D_b) = d(-D_{b'}) + 1$.

Then we have

$$\begin{aligned} d(D_b) + d(-D_b) &= c(D), \\ d(D_b) + d(-D_{b'}) + 1 &= c(D), \\ a(D) + a(D) &\leq c(D) - 1, \\ a(D) &\leq \frac{1}{2}(c(D) - 1). \end{aligned}$$

□

Lemma 4.2. *If $a(D) = \frac{1}{2}(c(D) - 1)$, then D is alternating.*

Proof. Suppose that D is not alternating. Then, D contains a portion of two successive overcrossings. Let b and b' be the base points of D just before and after two overcrossings as illustrated in Figure 15.

By Lemma 2.6, we have

$$\begin{aligned} d(D_b) + d(-D_b) &= c(D), \\ d(D_b') + d(-D_b') &= c(D). \end{aligned}$$

Then, we have

$$d(D_b) + d(-D_b) + d(D_b') + d(-D_b') = 2c(D). \tag{1}$$

By Lemma 2.4, we have

$$d(D_b') = d(D_b) + 2, \tag{2}$$

$$d(-D_b') = d(-D_b) + 2. \tag{3}$$

From the equations (1), (2), and (3), we have

$$\begin{aligned} d(D_b) + d(-D_b) + d(D_b) + d(-D_b) + 4 &= 2c(D), \\ d(D_b) + d(-D_b) &= c(D) - 2, \end{aligned}$$

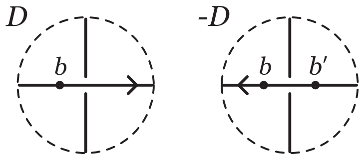


Figure 14.

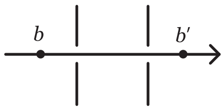


Figure 15.

$$a(D) + a(D) \leq c(D) - 2,$$
$$a(D) = \frac{1}{2}(c(D) - 2).$$

It contradicts the assumption. □

We say that two chords are *consecutive* if an endpoint of one chord is adjacent to an endpoint of the other chord as illustrated in Figure 16.

LEMMA 4.3. *For a chord diagram C, if any two consecutive chords intersect, then any two chords of C intersects.*

PROOF. Suppose that there exist two chords c_1 and c_2 such that they do not intersect. Let $\overline{c_1c_2}$ be a component of $C \setminus (c_1 \cup c_2)$ such that the endpoints of $\overline{c_1c_2}$ consist of an endpoint of c_1 and an endpoint of c_2. If there is no chord one of whose endpoint is on $\overline{c_1c_2}$, it is a contradiction. If there is a chord c' one of whose endpoint is on $\overline{c_1c_2}$, then c' does not intersect at least one of c_1 and c_2. Suppose, without loss of generality, that c' does not intersect c_1, we take c' as c_2. Then, we can obtain new $\overline{c_1c_2}$ such that the number of chords one of whose endpoints is on it is less than before. By repeating the argument above, we can find consecutive chords c_1 and c_2 such that $c_1 \cap c_2 = \emptyset$. It is a contradiction. □

LEMMA 4.4. *For a chord diagram C, if any two consecutive chords intersect, then C is represented as in* Figure 17.

PROOF. By Lemma 4.3, any two chords intersect, and all chords intersect as illustrated in Figure 17. □

PROOF OF THEOREM 1.4. If K is the $(2,p)$-torus knot, then we have $u(K) = \frac{1}{2}(c(K) - 1)$ by Theorem 1.2. We have $a(K) = \frac{1}{2}(c(K) - 1)$ by the inequality $u(K) \leq a(K)$ and Proposition 1.1.

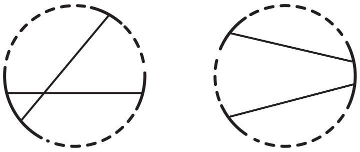


Figure 16.

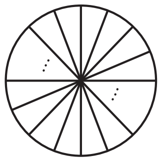


Figure 17.

Assume that $a(K)=\frac{1}{2}(c(K)-1)$. Let D be a diagram such that $c(D)=c(K)$. By the definition of the ascending number and Lemma 4.1, we have

$$a(K) \leq a(D) \leq \frac{1}{2}(c(D)-1) = \frac{1}{2}(c(K)-1).$$

Since $a(K) = \frac{1}{2}(c(K)-1)$, then $a(K) = a(D) = \frac{1}{2}(c(D)-1)$. Hence D is alternating by Lemma 4.2. Taking a base point just before an overcrossing, we have $d(D_b)=a(D)=a(K)$ by Lemma 2.5. Furthermore we note that $a(D)=a(-D)=\frac{C(D)-1}{2}$ by Lemma 2.6.

Let c_1 and c_2 be two successive crossings of D. We assign an orientation to D and a base point just before these crossings so that we first encounter an overcrossing. If c_1 and c_2 correspond to chords of type 1, then we can decrease the ascending number by a move for a chord of type 1. Hence we have $a(K) \leq \frac{1}{2}(c(K)-2)$. It contradicts the assumption. Therefore, any two consecutive chords of D intersect. By Lemma 4.3, any two chords of D intersect. Then we have a chord diagram of D as illustrated in Figure 17, by Lemma 4.4. The alternating knot diagram corresponding to such a chord diagram is a diagram of the $(2,p)$-torus knot. □

5. The ascending number of a certain family of knots

Ozawa [8] determines the ascending numbers of torus knots.

Theorem 5.1 ([8]). *Let p and q be coprime integers, and $T_{(p,q)}$ the (p,q)-torus knot. Then, we have*

$$a(T_{(p,q)}) = \frac{1}{2}(p-1)(q-1).$$

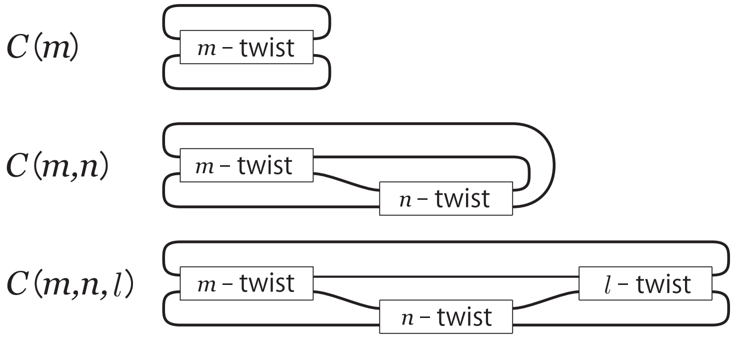


Figure 18.

In this section, we consider the families of 2-bridge knots as illustrated in Figure 18. Let $C(m)$, $C(m,n)$, and $C(m,n,l)$ be 2-bridge knots represented by Conway notation m, mn, and mnl, respectively, where m, n, and l are positive integers.

Theorem 5.2.

(1) $a(C(m)) = \frac{1}{2}(m-1)$ *if* m *is odd.*
(2) $a(C(m,n)) = \frac{1}{2}m$ *if* m *is even and* n *is odd integer.*
(3) $a(C(m,n,l)) = \frac{1}{2}(m+l-1)$ *if* m *and* n *are even and* l *is odd integer.*

Murasugi [7] defines the *signature* $\sigma(K)$ of a knot K as the signature of the matrix $S_K + S_K^T$, where S_K is the Seifert matrix of K and S_K^T is the transposed matrix of S_K. The following is obtained by the well-known result between the signature and the unknotting number, and the definition of the ascending number.

Lemma 5.3. *Let* K *be a knot. We have* $\frac{|\sigma(K)|}{2} \leq u(K) \leq a(K)$.

Proof of Theorem 5.2. (1) Since $C(m)$ is the torus knot of type $(2,m)$, $a(K) = \frac{1}{2}(m-1)$ by Theorem 5.1.

(2) Suppose that m is even and n is odd. We have $\sigma(K) = -m$. Hence $\frac{m}{2} \leq a(C(m,n))$ by Lemma 5.3. We obtain a diagram D of $C(m,n)$ with $a(D) = \frac{m}{2}$ by a whirl move for the portion of a diagram corresponding to n-twists. Therefore, we have $a(C(m,n)) = \frac{m}{2}$.

(3) Suppose that m and n are even and l is odd. We have $\sigma(K) = m + r - 1$. Hence $\frac{m+r-1}{2} \leq a(C(m,n,l))$ by Lemma 5.3. We obtain a diagram of $C(m,n,l)$ with $a(K) = \frac{m+r-1}{2}$ by a whirl move for the portion of a diagram corresponding to n-twists. Therefore, we have $a(m,n,r) = \frac{m+r-1}{2}$. □

Remark 5.4.

(1) $u(C(m)) = a(C(m)) = \frac{1}{2}(m-1)$ if n is odd.
(2) $u(C(m,n)) = a(C(m,n)) = \frac{1}{2}m$ if m is odd and n is even.
(3) $u(C(m,n,l)) = a(C(m,n,l)) = \frac{1}{2}(m+l-1)$ if m and n are even and l is odd.

Theorem 5.5. *Let n be an integer. If $n=1$ or $|n|\geq 3$, then $a(C(4,n))=2$.*

Proof. We denote $C(4,n)$ by K_n. If n is positive and odd, then $a(K_n)=2$ by Theorem 5.2 (2).

Suppose that n is even, or negative odd. We have $a(K_n)\leq 2$ by a whirl move for the portion of a diagram corresponding to n-half twists. We can show that K_n is not a twist knot by a knot invariant (for example, Conway polynomial, Jones polynomial), hence $a(K_n)\geq 2$ by Theorem 1.6. □

Remark 5.6.

(1) $a(K)=1$ for $n=-2,-1,2$.
(2) $a(K)=0$ for $n=0$.

Problem 5.7.

(1) Determine $a(C(m,n))$ when m and n are even ($n\neq 4$).
(2) Determine $a(C(m,n,r))$ when m, n, and r are odd.
(3) Determine $a(C(m,n,r))$ when m is even, and n and r are odd.

6. The ascending number of a reduced alternating diagram

According to Lemma 2.5, the warping degree of an alternating diagram takes two values $d(D)$ or $d(D)+1$ depending on a choice of base points.

Problem 6.1. For an alternating diagram, what property of a diagram does the ascending number depend on?

When a diagram is not reduced, we can make a diagram whose ascending number is arbitrarily large. We consider the ascending number of a knot represented by reduced alternating diagrams. Note that any two diagrams of an alternating knot is mutually transformed by a finite sequence of flype moves (Figure 19), that is, Tait's conjecture, see [6].

Let D_b be a reduced alternating diagram as illustrated in the left of Figure 20, and suppose that $d(D)=d(D_b)=n$. Then there are two warping crossings in the portion of D illustrated by bold lines. Let $D'_{b'}$ be a diagram obtained by the flype move from D, see the right of Figure 20. Then there is one warping crossing

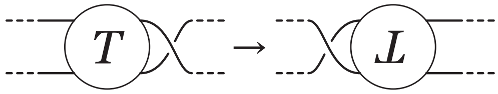


Figure 19.

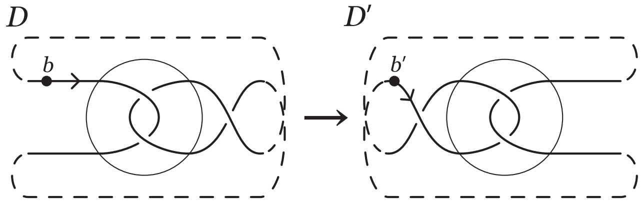


Figure 20.

in the portion of bold line, and the portions of D illustrated by dotted lines are not changed before a flype move. Hence

$$d(D') = d(D'_{b'}) = n - 1.$$

As above, we can decrease the ascending number of a certain diagram by a flype move while keeping a diagram reduced alternating.

Example 6.2. Let D be a diagram of 9_{11} as illustrated in Figure 21. Then, $d(D) = 4$. By a flype move to the portion denoted by a dotted circle, we have a diagram D' with $d(D') = 3$.

Problem 6.3. For given n, find the minimal value of $d(D)$ for a reduced alternating diagram D with $c(D) = n$.

We can obtain a diagram whose crossing number is arbitrary large and ascending number is small while keeping a diagram reduced alternating. For an alternating knot K, we denote $a_{max}(K)$ (resp. $a_{min}(K)$) the maximal (resp. minimal) of the ascending number among of all reduced alternating diagram of K.

Theorem 6.4. *For any natural number n, we have a knot such that $a_{max}(K) - a_{min}(K) \geq n$.*

Proof. Let K be a knot obtained by a connected sum of n copies of 4_1 knots and D a diagram of K as illustrated in the left of Figure 22. Then, $a(D) = 2n$. Let D' be a diagram of K obtained by n flype moves from D as illustrated in the right of Figure 22. Then, we have $a(D') = n$, and $a_{max}(K) - a_{min}(K) \geq a(D) - a(D') = n$. □

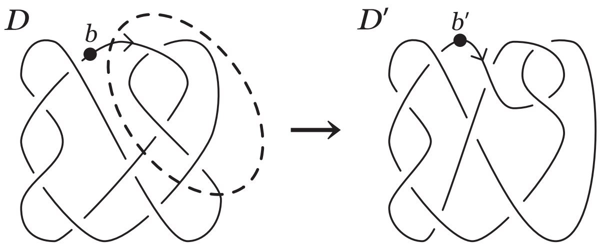


Figure 21.

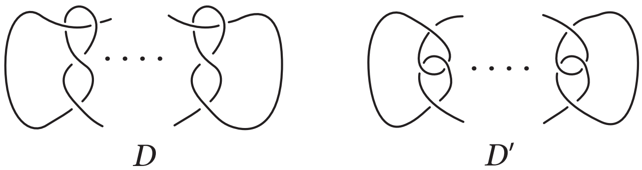


Figure 22.

7. Table of ascending number of knots

We give a table of the ascending number and unknotting number of prime knots with ten crossings or less. The notation for knots follows the Rolfsen's knot table [9]. The following points should be noted. The knots 10_{161} and 10_{162} in [9] are identical. So we give a numbering for 10_n ($n = 162, 163, 164, 165$) for the knot 10_{n+1} in [9]. The figures 10_{83} and 10_{86} in [9] should be interchanged. The lower bound of the ascending number is given by Theorem 1.6 and Lemma 5.3. The notation twist, 1, 2, . . . , 6, or $*$ in the fourth column indicates a method to determine the ascending number as follows. The notation "twist" means a twist knot. The upper bound is given by the transformations in Section 3. The number 1, 2, . . . , or 6 means that the upper bound of $a(K)$ is given by the corresponding move 1, 2, . . . , or 6 in Section 3. The notation $*$ means that it determined by the result of the first author [2]. The upper bounds of $a(10_{31})$ and $a(10_{58})$ are determined by the move in Example 3.2. The upper bounds of $a(10_{132})$ and $a(10_{136})$ are determined by the move in Figure 13. These informations are written in the fourth column.

We use D a knot diagram based on Rolfsen's knot table. For knots with unknown ascending number, we give by the set of possible values. For example, $(2, 3)$ means 2 or 3.

We remark that the authors think $a(10_9) = (2, 3, 4)$ but $(2, 3)$ in Jablan's table [3].

knot	$u(K)$	$a(K)$	$a(D)$	
3_1	1	1	1	twist
4_1	1	1	1	twist
5_1	2	2	2	
5_2	1	1	2	twist
6_1	1	1	2	twist
6_2	1	2	2	1
6_3	1	2	2	1
7_1	3	3	3	
7_2	1	1	3	twist
7_3	2	2	3	1
7_4	2	2	3	1
7_5	2	2	3	1
7_6	1	2	3	1
7_7	1	2	2	
8_1	1	1	3	twist
8_2	2	3	3	$*$
8_3	2	2	3	1
8_4	2	2	3	1
8_5	2	3	3	$*$
8_6	2	2	3	1
8_7	1	$(2,3)$	3	
8_8	2	2	3	1
8_9	1	$(2,3)$	3	
8_{10}	2	$(2,3)$	3	
8_{11}	1	2	3	1
8_{12}	2	2	3	1
8_{13}	1	2	3	1
8_{14}	1	2	3	1
8_{15}	2	2	3	1
8_{16}	2	2	3	1
8_{17}	1	2	3	1
8_{18}	2	2	2	1
8_{19}	3	3	3	
8_{20}	1	2	2	
8_{21}	1	2	2	
9_1	4	4	4	
9_2	1	1	4	twist
9_3	3	3	4	1
9_4	2	2	4	2
9_5	2	2	4	2
9_6	3	3	4	1
9_7	2	2	4	2

knot	$u(K)$	$a(K)$	$a(D)$	
9_8	2	2	4	2
9_9	3	3	4	1
9_{10}	3	3	4	1
9_{11}	2	3	4	1, $*$
9_{12}	1	2	4	2
9_{13}	3	3	4	1
9_{14}	1	2	3	1
9_{15}	2	2	4	2
9_{16}	3	3	4	1
9_{17}	2	$(2,3)$	3	
9_{18}	2	2	4	2
9_{19}	1	2	3	1
9_{20}	2	3	4	1, $*$
9_{21}	1	2	4	5
9_{22}	1	$(2,3)$	3	
9_{23}	2	2	4	2
9_{24}	1	$(2,3)$	3	
9_{25}	2	2	4	2
9_{26}	1	$(2,3)$	3	
9_{27}	1	$(2,3)$	4	1
9_{28}	1	$(2,3)$	4	1
9_{29}	2	$(2,3)$	4	1
9_{30}	1	$(2,3)$	4	1
9_{31}	2	$(2,3)$	3	
9_{32}	2	2	3	1
9_{33}	1	2	4	6
9_{34}	1	2	3	1
9_{35}	3	3	4	1
9_{36}	2	3	4	1, $*$
9_{37}	2	2	3	1
9_{38}	3	3	4	1
9_{39}	1	$(2,3)$	4	1
9_{40}	2	$(2,3)$	4	1
9_{41}	2	$(2,3)$	3	
9_{42}	1	2	2	
9_{43}	2	3	3	$*$
9_{44}	1	2	3	1
9_{45}	1	2	3	1
9_{46}	2	2	3	1
9_{47}	2	2	2	
9_{48}	2	2	2	
9_{49}	3	3	3	

knot	$u(K)$	$a(K)$	$a(D)$	
10_1	1	1	4	twist
10_2	3	4	4	$*$
10_3	2	2	4	2
10_4	2	2	4	2
10_5	2	$(3,4)$	4	$*$
10_6	3	3	4	1
10_7	1	2	4	2
10_8	2	3	4	1, $*$
10_9	1	$(2,3,4)$	4	
10_{10}	1	2	4	2
10_{11}	$(2,3)$	$(2,3)$	4	1
10_{12}	2	$(2,3)$	4	1
10_{13}	2	2	4	2
10_{14}	2	3	4	1, $*$
10_{15}	2	$(2,3)$	4	1
10_{16}	2	$(2,3)$	4	1
10_{17}	1	$(2,3,4)$	4	
10_{18}	1	2	4	2
10_{19}	2	$(2,3)$	4	1
10_{20}	2	2	4	2
10_{21}	2	3	4	1, $*$
10_{22}	2	$(2,3)$	4	1
10_{23}	1	$(2,3)$	4	1
10_{24}	2	2	4	2
10_{25}	2	3	4	1, $*$
10_{26}	1	$(2,3)$	4	1
10_{27}	1	$(2,3)$	4	1
10_{28}	2	$(2,3)$	4	1
10_{29}	2	$(2,3)$	4	1
10_{30}	1	$(2,3)$	4	1
10_{31}	1	2	4	Ex. 3.2
10_{32}	1	$(2,3)$	4	1
10_{33}	1	$(2,3)$	4	1
10_{34}	2	2	4	2
10_{35}	2	2	4	2
10_{36}	2	2	4	2
10_{37}	2	2	4	2
10_{38}	2	2	4	2
10_{39}	2	3	4	1, $*$
10_{40}	2	$(2,3)$	4	1
10_{41}	2	$(2,3)$	4	1
10_{42}	1	$(2,3)$	4	1

knot	$u(K)$	$a(K)$	$a(D)$	
10_{43}	2	$(2,3)$	4	1
10_{44}	1	$(2,3)$	4	1
10_{45}	2	$(2,3)$	3	1
10_{46}	3	4	4	$*$
10_{47}	$(2,3)$	$(3,4)$	4	
10_{48}	2	$(2,3,4)$	4	
10_{49}	3	3	4	1
10_{50}	2	3	4	1, $*$
10_{51}	$(2,3)$	$(2,3)$	4	1
10_{52}	2	$(2,3)$	4	1
10_{53}	3	3	4	1
10_{54}	$(2,3)$	$(2,3)$	4	1
10_{55}	2	2	4	2
10_{56}	2	3	4	1, $*$
10_{57}	2	$(2,3)$	4	1
10_{58}	2	2	4	Ex. 3.2
10_{59}	1	$(2,3)$	4	1
10_{60}	1	$(2,3)$	3	
10_{61}	$(2,3)$	3	4	1, $*$
10_{62}	2	$(2,3,4)$	4	
10_{63}	2	2	4	2
10_{64}	2	$(2,3,4)$	4	
10_{65}	2	$(2,3)$	4	1
10_{66}	3	3	4	1
10_{67}	2	2	4	2
10_{68}	2	$(2,3)$	4	1
10_{69}	2	$(2,3)$	4	1
10_{70}	2	$(2,3)$	4	1
10_{71}	1	$(2,3)$	4	1
10_{72}	2	3	4	1, $*$
10_{73}	1	$(2,3)$	4	1
10_{74}	2	$(2,3)$	4	1
10_{75}	2	$(2,3)$	3	
10_{76}	$(2,3)$	3	4	1, $*$
10_{77}	$(2,3)$	$(2,3)$	4	1
10_{78}	2	$(2,3)$	4	1
10_{79}	$(2,3)$	$(2,3,4)$	4	
10_{80}	3	3	4	1
10_{81}	2	$(2,3)$	4	1
10_{82}	1	$(2,3)$	4	1
10_{83}	2	2	4	5
10_{84}	1	$(2,3)$	4	1

knot	$u(K)$	$a(K)$	$a(D)$	
10_{85}	2	3	4	1, $*$
10_{86}	2	2	4	4
10_{87}	2	$(2,3)$	4	1
10_{88}	1	2	4	4
10_{89}	2	2	3	1
10_{90}	2	$(2,3)$	4	1
10_{91}	1	$(2,3)$	4	1
10_{92}	2	3	4	1, $*$
10_{93}	2	$(2,3)$	4	1
10_{94}	2	$(2,3)$	4	1
10_{95}	1	$(2,3)$	4	1
10_{96}	2	$(2,3)$	4	1
10_{97}	2	$(2,3)$	4	1
10_{98}	2	3	4	1, $*$
10_{99}	2	$(2,3,4)$	4	
10_{100}	$(2,3)$	3	4	1, $*$
10_{101}	3	3	4	1
10_{102}	1	$(2,3)$	4	1
10_{103}	3	3	4	1
10_{104}	1	$(2,3)$	4	1
10_{105}	2	$(2,3)$	4	1
10_{106}	2	$(2,3)$	4	1
10_{107}	1	$(2,3)$	4	1
10_{108}	2	$(2,3)$	4	1
10_{109}	2	$(2,3,4)$	4	
10_{110}	2	$(2,3)$	4	1
10_{111}	2	3	4	1, $*$
10_{112}	2	$(2,3)$	3	
10_{113}	1	$(2,3)$	3	
10_{114}	1	2	3	1
10_{115}	2	2	4	2
10_{116}	2	$(2,3)$	4	1
10_{117}	2	$(2,3)$	4	1
10_{118}	1	$(2,3)$	4	1
10_{119}	1	$(2,3)$	4	1
10_{120}	3	3	4	1
10_{121}	2	$(2,3)$	4	1
10_{122}	2	$(2,3)$	3	
10_{123}	2	$(2,3)$	3	
10_{124}	4	4	4	
10_{125}	2	$(2,3)$	3	
10_{126}	2	$(2,3)$	3	

knot	$u(K)$	$a(K)$	$a(D)$	
10_{127}	2	3	3	$*$
10_{128}	3	3	4	1
10_{129}	1	$(2,3)$	3	
10_{130}	2	$(2,3)$	3	
10_{131}	1	$(2,3)$	3	
10_{132}	1	2	3	Fig. 13
10_{133}	1	2	3	1
10_{134}	3	3	3	
10_{135}	2	2	3	1
10_{136}	1	2	3	Fig. 13
10_{137}	1	2	3	1
10_{138}	2	$(2,3)$	3	
10_{139}	4	4	4	
10_{140}	2	2	3	1
10_{141}	1	$(2,3)$	3	
10_{142}	3	3	4	1
10_{143}	1	$(2,3)$	3	
10_{144}	2	2	3	1
10_{145}	2	2	3	1
10_{146}	1	2	3	1
10_{147}	1	2	3	1
10_{148}	2	$(2,3)$	3	
10_{149}	2	3	3	$*$
10_{150}	2	3	3	$*$
10_{151}	2	$(2,3)$	3	
10_{152}	4	4	4	
10_{153}	2	$(2,3)$	4	1
10_{154}	3	3	4	1
10_{155}	2	2	3	1
10_{156}	1	$(2,3)$	3	
10_{157}	2	2	3	1
10_{158}	2	2	3	1
10_{159}	1	2	2	
10_{160}	2	2	2	
10_{161}	3	3	3	
10_{162}	2	$(2,3)$	3	
10_{163}	2	2	2	
10_{164}	1	$(2,3)$	3	
10_{165}	2	$(2,3)$	3	

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