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A COMBINATORIAL PROOF OF THE WEIGHTED SUM FORMULA FOR FINITE AND SYMMETRIC MULTIPLE ZETA(-STAR) VALUES

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Abstract

Hirose, Saito, and the author established the weighted sum formula for finite multiple zeta(-star) values. In this paper, we present its alternative proof. The proof is also valid for symmetric multiple zeta(-star) values.

1. Introduction

For positive integers $k_1, \ldots, k_r$ with $k_r \geq 2$, the multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by

$$\zeta(k_1, \ldots, k_r) := \sum_{0 < n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \in \mathbb{R},$$

$$\zeta^\star(k_1, \ldots, k_r) := \sum_{0 < n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \in \mathbb{R}.$$

We set a $\mathbb{Q}$-algebra $\mathcal{A}$ by

$$\mathcal{A} := \left(\prod_p \mathbb{Z}/p\mathbb{Z} \right) \Big/ \left(\bigoplus_p \mathbb{Z}/p\mathbb{Z} \right),$$

where p runs over all primes. For positive integers $k_1, \ldots, k_r$, the finite multiple zeta values (FMZVs) and the finite multiple zeta-star values (FMZSVs) are defined by

$$\zeta_\mathcal{A}(k_1, \ldots, k_r) := \left(\sum_{0 < n_1 < \cdots < n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p \right)_p \in \mathcal{A},$$

$$\zeta_\mathcal{A}^\star(k_1, \ldots, k_r) := \left(\sum_{0 < n_1 \leq \cdots \leq n_r < p} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}} \bmod p \right)_p \in \mathcal{A}.$$

The symmetric multiple zeta values (SMZVs) were introduced by Kaneko–Zagier [8]. For positive integers $k_1, \ldots, k_r$, we define

$$\zeta_{\mathcal{S}}^*(k_1, \ldots, k_r) := \sum_{i=0}^{r} (-1)^{k_{i+1} + \cdots + k_r} \zeta^*(k_1, \ldots, k_i) \zeta^*(k_r, \ldots, k_{i+1}) \in \mathbb{R}.$$

Here, the symbol ζ^* on the right-hand side means the regularized value coming from the harmonic regularization, i.e., a real value obtained by taking constant terms of harmonic regularization as explained in Ihara–Kaneko–Zagier [4]. In the sum, we understand $\zeta^*(\emptyset) = 1$. Let $\mathcal{Z}_{\mathbb{R}}$ be the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by 1 and all MZVs, which is a $\mathbb{Q}$-algebra. Then the SMZVs are defined by

$$\zeta_{\mathcal{S}}(k_1, \ldots, k_r) := \zeta_{\mathcal{S}}^*(k_1, \ldots, k_r) \bmod \zeta(2) \in \mathcal{Z}_{\mathbb{R}}/(\zeta(2)).$$

For positive integers $k_1, \ldots, k_r$, we also define the symmetric multiple zeta-star values (SMZSVs) by

$$\zeta_{\mathcal{S}}^{\star}(k_1, \ldots, k_r) := \sum_{\substack{\square \text{ is either a comma ``,''} \\ \text{or a plus ``+''}}} \zeta_{\mathcal{S}}^*(k_1 \square \cdots \square k_r) \bmod \zeta(2) \in \mathcal{Z}_{\mathbb{R}}/(\zeta(2)).$$

Let $\mathcal{Z}_A$ denote the $\mathbb{Q}$-vector subspace of $\mathcal{A}$ spanned by 1 and all FMZVs. Kaneko and Zagier conjecture that there is an isomorphism between $\mathcal{Z}_A$ and $\mathcal{Z}_{\mathbb{R}}/(\zeta(2))$ as $\mathbb{Q}$-algebras such that $\zeta_{\mathcal{A}}(k_1, \ldots, k_r)$ and $\zeta_{\mathcal{S}}(k_1, \ldots, k_r)$ correspond with each other (for more details, see Kaneko–Zagier [7, 8]). In the following, we use the letter $\mathcal{F}$ stands for either $\mathcal{A}$ or $\mathcal{S}$, e.g., the symbol $\zeta_{\mathcal{F}}$ means $\zeta_{\mathcal{A}}$ or $\zeta_{\mathcal{S}}$.

Hirose, Saito, and the author [2] proved the following weighted sum formula for FMZ(S)Vs. In this paper, we will give its alternative proof. Our proof is also valid for SMZ(S)Vs.

THEOREM 1.1. *Let k be a positive integer, r a positive odd integer, and i an integer with $1 \le i \le r \le k$. Then we have*

$$\sum_{\substack{k_1 + \cdots + k_r = k \\ k_1, \ldots, k_r \ge 1}} 2^{k_i} \zeta_{\mathcal{F}}(k_1, \ldots, k_r) = 0,$$

$$\sum_{\substack{k_1 + \cdots + k_r = k \\ k_1, \ldots, k_r \ge 1}} 2^{k_i} \zeta_{\mathcal{F}}^{\star}(k_1, \ldots, k_r) = 0.$$

REMARK 1.2. We note that similar weighted sum formulas for MZVs are known (see Guo–Xie [1], Ohno–Zudilin [10], and Ong–Eie–Liaw [11]). Kamano [6] also obtained somewhat different weighted sum formulas for FMZVs.

2. Proof of the main theorem

2.1. Notation. An index is a sequence of positive integers, and we denote by $\mathcal{I}$ the $\mathbb{Q}$-linear space spanned by the indices. Maps defined for indices, such as $\zeta_{\mathcal{F}}$, will be extended $\mathbb{Q}$-linearly. For an index $\boldsymbol{k} = (k_1, \ldots, k_r)$, the integer $k := k_1 + \cdots + k_r$ is called the weight of $\boldsymbol{k}$ (denoted by $\mathrm{wt}(\boldsymbol{k})$) and the integer r is called the depth of $\boldsymbol{k}$ (denoted by $\mathrm{dep}(\boldsymbol{k})$). We write $(\{1\}^m) = (\underbrace{1, \ldots, 1}_{m})$ for a nonnegative integer m. For indices $\boldsymbol{k} = (k_1 \ldots, k_r)$ and $\boldsymbol{k}' = (k_1', \ldots, k_r')$ of the same depths, the symbol $\boldsymbol{k} \oplus \boldsymbol{k}'$ represents the componentwise sum, i.e., $\boldsymbol{k} \oplus \boldsymbol{k}' := (k_1 + k_1', \ldots, k_r + k_r')$.

DEFINITION 2.1. For an index $\boldsymbol{k} = (k_1 \ldots, k_r)$, we define $\phi(\boldsymbol{k})$ by

$$\phi(\boldsymbol{k}) := (-1)^r \sum_{\substack{\square \text{ is either a comma ``,''} \\ \text{or a plus ``+''}}} (\underbrace{1 \,\square\, 1 \,\square\, \cdots \,\square\, 1}_{\substack{\text{the number of} \\ \text{``1'' is } k_1}}, \ldots, \underbrace{1 \,\square\, 1 \,\square\, \cdots \,\square\, 1}_{\substack{\text{the number of} \\ \text{``1'' is } k_r}}) \in \mathcal{I}.$$

For example, we have $\phi(1, 2, 2) = -(1, 2, 2) - (1, 1, 1, 2) - (1, 2, 1, 1) - (1, 1, 1, 1, 1)$.

DEFINITION 2.2. For a nonempty index $\boldsymbol{k} = (k_1, \ldots, k_r)$, we define Hoffman's dual index of $\boldsymbol{k}$ by

$$\boldsymbol{k}^{\vee} = (\underbrace{1, \ldots, 1}_{k_1} + \underbrace{1, \ldots, 1}_{k_2} + 1, \ldots, 1 + \underbrace{1, \ldots, 1}_{k_r}).$$

For positive integers k, r, i with $1 \le i \le r \le k$, we set

$$F(k, r, i) := \sum_{\substack{k_1 + \cdots + k_r = k \\ k_1, \ldots, k_r \ge 1}} 2^{k_i - 1} \cdot (k_1, \ldots, k_r) \in \mathcal{I}.$$

Throughout this paper, we always assume that $\boldsymbol{e}$ runs over sequences of nonnegative integers. (Recall that an index is a sequence of positive integers.) For an index $\boldsymbol{k}$ and a nonnegative integer l, we also set

$$G_1(\boldsymbol{k}, l) := \sum_{\substack{\mathrm{wt}(\boldsymbol{e}) = l \\ \mathrm{dep}(\boldsymbol{e}) = \mathrm{dep}(\boldsymbol{k})}} (\boldsymbol{k} \oplus \boldsymbol{e}) \in \mathcal{I},$$

$$G_2(\boldsymbol{k}, l) := \sum_{\substack{\mathrm{wt}(\boldsymbol{e}) = l \\ \mathrm{dep}(\boldsymbol{e}) = \mathrm{dep}(\boldsymbol{k}^{\vee})}} (\boldsymbol{k}^{\vee} \oplus \boldsymbol{e})^{\vee} \in \mathcal{I},$$

$$G(\boldsymbol{k}, l) := G_1(\boldsymbol{k}, l) - G_2(\boldsymbol{k}, l) \in \mathcal{I}.$$

For example, we have $G_1((2,3),1) = (3,3) + (2,4)$. For positive integers k, r, i with $1 \le i \le r \le k$, we put

$$
\begin{aligned}
&H(k,r,i) \\
&:= F(k,r,i) \\
&\quad - \left(\sum_{l=1}^{k-r-1} 2^{l-1} G(({\{1\}}^{i-1}, l+1, {\{1\}}^{r-i}), k-r-l) + G(({\{1\}}^r), k-r) \right) \in \mathcal{I}.
\end{aligned}
$$

2.2. Proof of Theorem 1.1. To prove Theorem 1.1, we use Theorems 2.3 and 2.4, and Lemma 2.5.

THEOREM 2.3 (Hoffman [3], Jarossay [5]). *For an index $\boldsymbol{k}$, we have*

$$
\zeta_{\mathcal{F}}(\boldsymbol{k}) = \zeta_{\mathcal{F}}(\phi(\boldsymbol{k})).
$$

The following relation for FMZVs and SMZVs was conjectured by Kaneko and established by Oyama [12].

THEOREM 2.4 (Oyama [12]). *For an index $\boldsymbol{k}$ and a nonnegative integer l, we have*

$$
\zeta_{\mathcal{F}}(G(\boldsymbol{k}, l)) = 0.
$$

LEMMA 2.5 (Key lemma). *Let k be a positive integer, r a positive odd integer, and i an integer with $1 \le i \le r \le k$. Then we have*

$$
H(k,r,i) + \phi(H(k,r,i)) = \begin{cases} ({\{1\}}^k) & (k\text{: even}), \\ 0 & (k\text{: odd}). \end{cases}
$$

To prove Lemma 2.5, we need Lemmas 2.6 and 2.7.

LEMMA 2.6. *Let k, r be positive integers and i an integer with $1 \le i \le r \le k$. Then we have*

$$
F(k,r,i) - \left(\sum_{l=1}^{k-r-1} 2^{l-1} G_1(({\{1\}}^{i-1}, l+1, {\{1\}}^{r-i}), k-r-l) + G_1(({\{1\}}^r), k-r) \right)
$$
$$
= 2^{k-r-1}({\{1\}}^{i-1}, k-r+1, {\{1\}}^{r-i}).
$$

PROOF. We note that the depths of all the indices on the left-hand side are r. Put

$$
A := \sum_{l=1}^{k-r-1} 2^{l-1} G_1(({\{1\}}^{i-1}, l+1, {\{1\}}^{r-i}), k-r-l) + G_1(({\{1\}}^r), k-r).
$$

Fix an index $(a_1, \ldots, a_i, \ldots, a_r)$ of weight k and depth r. Since the number of indices $(a_1, \ldots, a_i, \ldots, a_r)$ included in A is

$$
\begin{cases}
\displaystyle\sum_{l=1}^{a_i-1} 2^{l-1} + 1 = 2^{a_i-1} & (1 \le a_i \le k - r), \\[2ex]
\displaystyle\sum_{l=1}^{k-r-1} 2^{l-1} + 1 = 2^{k-r-1} & (a_i = k - r + 1),
\end{cases}
$$

we find that all the indices appeared in $F(k, r, i)$ and A are same except for $2^{k-r-1}(\{1\}^{i-1}, k - r + 1, \{1\}^{r-i})$. Thus we find the result. $\qquad\square$

LEMMA 2.7. *Put* $\Phi(I) := I + \phi(I)$ *for* $I \in \mathcal{I}$. *Let* k *be a positive integer,* r *a positive odd integer, and* i *an integer with* $1 \le i \le r \le k$. *Then we have*

$$
\Phi\left(\sum_{l=1}^{k-r-1} 2^{l-1} G_2((\{1\}^{i-1}, l+1, \{1\}^{r-i}), k-r-l) + G_2((\{1\}^r), k-r)\right)
$$

$$
= \begin{cases}
-2^{k-r-1}\Phi(\{1\}^{i-1}, k-r+1, \{1\}^{r-i}) + (\{1\}^k) & (k:\ even), \\
-2^{k-r-1}\Phi(\{1\}^{i-1}, k-r+1, \{1\}^{r-i}) & (k:\ odd).
\end{cases}
$$

PROOF. Put

$$
B := \sum_{l=1}^{k-r-1} 2^{l-1} G_2((\{1\}^{i-1}, l+1, \{1\}^{r-i}), k-r-l) + G_2((\{1\}^r), k-r),
$$

$$
C := -2^{k-r-1}\Phi(\{1\}^{i-1}, k-r+1, \{1\}^{r-i}).
$$

Then we have

$$
B = \sum_{l=1}^{k-r-1} 2^{l-1} \sum_{\substack{\mathrm{wt}(\boldsymbol{e})=k-r-l \\ \mathrm{dep}(\boldsymbol{e})=l+1}} (((i, \{1\}^{l-1}, r-i+1) \oplus \boldsymbol{e})^\vee) + (\{1\}^k)
$$

$$
= \sum_{l=1}^{k-r-1} 2^{l-1} \sum_{\substack{k_1+\cdots+k_{k-r-l+1}=k-r+1 \\ k_1,\ldots,k_{k-r-l+1}\ge 1}} (\{1\}^{i-1}, k_1, \ldots, k_{k-r-l+1}, \{1\}^{r-i}) + (\{1\}^k).
$$

Fix an index $(\{1\}^{i-1}, a_1, \ldots, a_d, \{1\}^{r-i})$ of weight k and depth $d + r - 1$ with $2 \le d \le k - r$. Then the numbers of indices $(\{1\}^{i-1}, a_1, \ldots, a_d, \{1\}^{r-i})$ included in B and $\phi(B)$ are, respectively,

$$
2^{k-r-d} \quad \text{and} \quad \sum_{l=k-r-d+1}^{k-r-1} 2^{l-1} \cdot (-1)^{k-l} \cdot \binom{d-1}{k-r-l}.
$$

We have

$$2^{k-r-d} + \sum_{l=k-r-d+1}^{k-r-1} 2^{l-1} \cdot (-1)^{k-l} \cdot \binom{d-1}{k-r-l}$$

$$= 2^{k-r-d} + 2^{k-r-d} \cdot (-1)^{r+d+1} \cdot \sum_{l=0}^{d-2} (-2)^l \binom{d-1}{l}$$

$$= 2^{k-r-d} \cdot (-1)^d \cdot \left((-1)^d + \sum_{l=0}^{d-2} (-2)^l \binom{d-1}{l} \right) \quad \text{(by } r\text{: odd)}.$$

Since

$$(1 - 2x)^n = \sum_{a=0}^{n} (-2x)^a \cdot \binom{n}{a},$$

we have

$$\sum_{l=0}^{d-2} (-2)^l \binom{d-1}{l} = (-1)^{d-1} - (-2)^{d-1}$$

by substituting $x = 1$, $a = l$, and $n = d - 1$. Thus we get

$$2^{k-r-d} + \sum_{l=k-r-d+1}^{k-r-1} 2^{l-1} \cdot (-1)^{k-l} \cdot \binom{d-1}{k-r-l}$$

$$= 2^{k-r-d} \cdot (-1)^d \cdot ((-1)^d + (-1)^{d-1} - (-2)^{d-1})$$

$$= 2^{k-r-1}.$$

Similarly, the number of indices $(\{1\}^k)$ in $B + \phi(B)$ is

$$1 + \sum_{l=1}^{k-r-1} 2^{l-1} \cdot (-1)^{k-l} \cdot \binom{k-r}{k-r-l} + (-1)^k$$

$$= 1 + \frac{1}{2}(-1)^k \sum_{l=1}^{k-r-1} (-2)^l \binom{k-r}{l} + (-1)^k$$

$$= 1 + \frac{1}{2}(-1)^k ((-1)^{k-r} - 1 - (-2)^{k-r}) + (-1)^k$$

$$= 2^{k-r-1} + \frac{1}{2}(1 + (-1)^k) \quad \text{(by } r\text{: odd)}$$

$$= \begin{cases} 2^{k-r-1} + 1 & (k\text{: even}), \\ 2^{k-r-1} & (k\text{: odd}). \end{cases}$$

Then we find

$$
\begin{aligned}
&B + \phi(B) \\
&= \sum_{d=2}^{k-r} \sum_{\substack{a_1+\cdots+a_d=k-r+1 \\ a_1,\ldots,a_d \geq 1}} 2^{k-r-1}(\{1\}^{i-1}, a_1, \ldots, a_d, \{1\}^{r-i}) \\
&\quad + \begin{cases} (2^{k-r-1}+1)(\{1\}^k) & (k\colon \text{even}), \\ 2^{k-r-1}(\{1\}^k) & (k\colon \text{odd}). \end{cases}
\end{aligned}
$$

On the other hand, by the direct calculation, we have

$$
C = \sum_{d=2}^{k-r+1} \sum_{\substack{a_1+\cdots+a_d=k-r+1 \\ a_1,\ldots,a_d \geq 1}} 2^{k-r-1}(\{1\}^{i-1}, a_1, \ldots, a_d, \{1\}^{r-i}).
$$

This finishes the proof. $\qquad\square$

PROOF OF LEMMA 2.5. By Lemmas 2.6 and 2.7, we easily find the lemma holds. $\qquad\square$

PROOF OF THEOREM 1.1 (The first statement). By Theorems 2.3 and 2.4, and Lemma 2.5, we see the theorem holds. $\qquad\square$

Now we prove the second statement of Theorem 1.1. We can prove this in the same manner as in Hirose–Murahara–Saito [2]. The following formulas are well known (see, e.g., Sakugawa–Seki [13]).

LEMMA 2.8. *For positive integers k_1, ..., k_r, we have*

$$
\sum_{l=0}^{r} (-1)^l \zeta_{\mathcal{F}}^{\star}(k_1, \ldots, k_l) \zeta_{\mathcal{F}}(k_r, \ldots, k_{l+1}) = 0.
$$

Here, we understand $\zeta_{\mathcal{F}}(\emptyset) = \zeta_{\mathcal{F}}^{\star}(\emptyset) = 1$.

PROPOSITION 2.9 (Hoffman [3], Murahara [9]). *For positive integers k_1, ..., k_r, we have*

$$
\sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = 0,
$$

$$
\sum_{\sigma \in \mathfrak{S}_r} \zeta_{\mathcal{F}}^{\star}(k_{\sigma(1)}, \ldots, k_{\sigma(r)}) = 0,
$$

where $\mathfrak{S}_r$ is the symmetric group of degree r.

Proof of Theorem 1.1 (The second statement).   By Lemma 2.8, we have

$$\sum_{l=0}^{r}(-1)^l \sum_{\substack{k_1+\cdots+k_r=k \\ k_1,\ldots,k_r\geq 1}} 2^{k_i}\zeta_{\mathcal{F}}^{\star}(k_1,\ldots,k_l)\zeta_{\mathcal{F}}(k_r,\ldots,k_{l+1}) = 0.$$

By using the first statement of Theorem 1.1 and Proposition 2.9, we find the result.   □

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