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THE TOPOLOGY OF EXTRINSICALLY FLAT CLOSED STRIPS ON GIVEN KNOTS IN SPACES OF CONSTANT CURVATURE

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Abstract

In the author's previous work, the isotopy types of developable closed strips along arbitrarily given real-analytic knots in 3-dimensional spaceforms (i.e. simply-connected complete 3-spaces of constant curvature) were determined. In this paper, we generalize this to arbitrary 3-spaces of constant curvature, which might be non-simply-connected or non-orientable.

1. Introduction

Let M^3 be a 3-space of constant curvature (i.e. a 3-dimensional Riemannian manifold with constant sectional curvature) without boundary. A space curve in M^3 is called *regular* if its velocity vector never vanishes. A regular closed space curve is called a *knot* if it has no self-intersection. In this paper, we fix an arbitrarily given C^ω-knot (i.e. real-analytic knot) in M^3 and denote it by $\gamma=\gamma(s)\colon \mathbb{S}^1 \to M^3$, where $\mathbb{S}^1 := \mathbb{R}/l\mathbb{Z}$ and $l>0$ is constant. Let $\xi=\xi(s)\colon \mathbb{R} \longrightarrow TM^3$ be a C^ω-vector field along γ so that $\xi(s)$ is either l-odd-periodic (i.e. $\xi(s+l) = -\xi(s)$ for $s \in \mathbb{R}$) or l-periodic (i.e. $\xi(s+l) = \xi(s)$ for $s \in \mathbb{R}$). A ruled surface $F_{\gamma,\xi}\colon \mathbb{R} \times (-\epsilon, \epsilon) \to M^3$ given by

$$F_{\gamma,\xi}(s,u) := \mathrm{Exp}_{\gamma(s)}(u\xi(s)) \tag{1}$$

is called a *closed strip* (or a *strip*, for short) along γ if $\gamma'(s)$ and $\xi(s)$ are linearly independent for $s \in \mathbb{R}$, where $\epsilon > 0$ is sufficiently small and Exp_p denotes the exponential map at $p \in M^3$. The strip $F_{\gamma,\xi}$ is orientable (resp. non-orientable) if ξ is periodic (resp. odd-periodic).

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In Euclidean 3-space $\mathbb{R}^3$, a ruled surface is called *developable* if it is flat (i.e. has zero Gaussian curvature). The definition of developability can be generalized to M^3 as follows: A ruled strip $F_{\gamma,\xi}$ in M^3 is called *developable* if

$$\gamma' \wedge \xi \wedge D_s\xi = 0 \tag{2}$$

holds, where $\gamma' := d\gamma/ds$, the wedge is the exterior product on TM^3 and D is the Levi-Civita connection of M^3. The developability of a ruled surface in M^3 is equivalent to the extrinsic flatness of the surface $F_{\gamma,\xi}$, where *extrinsic flatness* means that the extrinsic curvature (i.e. the product of the two principal curvatures) vanishes identically.

In the case of M^3 being a spaceform (i.e. simply-connected complete space of constant curvature), the isotopy types of developable closed strips along γ were determined in the author's previous work [2]. So, it is natural to generalize [2] to 3-spaces M^3 of constant curvature, which might be non-simply-connected. However, in the proof of the main result in [2], the simple-connectedness of M^3 was used in an essential way. In fact, if M^3 is a spaceform, then it suffices to study only the case that M^3 is a 3-sphere $\mathbb{S}^3$ because a classically well-known central projection of $\mathbb{S}^3$ into $\mathbb{R}^3$ (resp. hyperbolic 3-space $\mathbb{H}^3$ into $\mathbb{R}^3$, called Klein's model) preserves the developability of surfaces in $\mathbb{S}^3$ (resp. $\mathbb{H}^3$). In this paper, without using such a projection and taking a more direct approach, we prove the generalization of the result in [2] to arbitrary 3-spaces M^3 of constant curvature. The key to this approach is to take a sufficiently small open tubular neighborhood Ω_γ of γ, and to consider the isotopy types of the developable closed strips in Ω_γ. The tubular neighborhood Ω_γ is called *type I* (resp. *type II*) if Ω_γ is orientable (resp. non-orientable). If Ω_γ is of type I, then we take one of the orientations of Ω_γ and fix that choice. The isotopy type of $F_{\gamma,\xi}$ in Ω_γ is determined by its *geometric twisting number* $\mathrm{Gtn}(F_{\gamma,\xi})$, which is defined in Sections 2 and 3 (cf. [2, page 346], [1]). We remark that the behavior of the isotopy types depends on the orientability of Ω_γ. In fact, $\mathrm{Gtn}(F_{\gamma,\xi})$ can be taken as

$$\mathrm{Gtn}(F_{\gamma,\xi}) \in \begin{cases} (1/2)\mathbb{Z}, & (\text{if } \Omega_\gamma \text{ is of type I}), \\ \{0, 1/2\} & (\text{if } \Omega_\gamma \text{ is of type II}). \end{cases} \tag{3}$$

Our main result is divided into the cases of type I and II, as follows: Let $Z_\kappa \subset \mathbb{S}^1$ be the zeros of the curvature function κ of γ, and τ the torsion function of γ defined on $\mathbb{S}^1 \setminus Z_\kappa$. We assume that there exists a point $s = a$ such that $\kappa(a) \neq 0$.

THEOREM A. *Suppose that Ω_γ is of type I and fix one of the orientations of Ω_γ.*

(I) *If $\tau(s)$ vanishes identically on $\mathbb{S}^1 \setminus Z_\kappa$, then a closed developable C^ω-strip (and even a C^∞-strip) along γ can only have zero geometric twisting number.*

(II) *If $\tau(s)$ changes sign, then for each $n \in (1/2)\mathbb{Z}$, there exists a closed developable C^ω-strip along γ whose geometric twisting number is n.*

(III) *If $\tau \leq 0$ on $\mathbb{S}^1 \setminus Z_\kappa$ and there exists a point $s_0 \in \mathbb{S}^1$ such that $\tau(s_0) < 0$, then*

(i) *for each $n > 0$ ($n \in (1/2)\mathbb{Z}$), there exists a closed developable C^ω-strip along γ whose geometric twisting number is n, and*

(ii) *if $n < 0$ ($n \in (1/2)\mathbb{Z}$), there is no closed developable C^∞-strip along γ whose geometric twisting number is n.*

In (III), *the statement with all of the four inequality signs reversed also holds.*

THEOREM B. *If Ω_γ is of type II, then for all $n \in \{0, 1/2\}$, there exists a closed developable C^ω-strip along γ whose geometric twisting number is n.*

With γ real-analytic, the cases that are stated not to exist in (I), (III)-(ii) of Theorem A will still not exist even if we allow the developable strip to be merely C^∞ (cf. [2], [1]). This is the meaning of the phrase in parentheses in (I) of Theorem A.

We remark that the cases of [2, Theorems 1.1 and 1.2] are contained in Theorem A. We show Theorems A and B in Sections 2 and 3, respectively. See [2] for detailed history and references on developable strips along γ.

2. The case of type I

In this section, we suppose that Ω_γ is orientable, i.e. of type I. We first explain the way to generalize the discussion of the preliminaries in [2] to M^3. Let s be an arc-length parameter of γ and $\langle\ ,\ \rangle$ the Riemannian metric tensor of M^3. The curvature function of γ is denoted by $\kappa(s) := |D_s\gamma'(s)|$, where $|v| := \sqrt{\langle v, v\rangle}$ for $v \in TM^3$. Although the Frenet frame $\{e := \gamma',\ n := D_s\gamma'/\kappa,\ b := e \times n\}$ (the cross '$\times$' is the vector product in Ω_γ) of γ cannot be defined at the zeros of κ, there exists a positive orthonormal C^ω-frame $\{e, \hat{n}, \hat{b}\}$ on $\mathbb{R}$ satisfying the modification of the Frenet–Serret formula

$$D_s e = \hat{\kappa}\hat{n}, \quad D_s\hat{n} = -\hat{\kappa}e + \hat{\tau}\hat{b}, \quad D_s\hat{b} = -\hat{\tau}\hat{n}, \tag{4}$$

where $\hat{n}$ and $\hat{b}$ are certain C^ω-vector fields, and $\hat{\kappa}$ and $\hat{\tau}$ are certain C^ω-functions (cf. [2, Equation (2.2)]). If $\kappa(s) \neq 0$, then we have $\kappa(s) = |\hat{\kappa}(s)|$ and

$$\hat{n}(s) = \delta(s)n(s), \quad \hat{b}(s) = \delta(s)b(s),$$

where $\delta := \hat{\kappa}/\kappa$ $(= \pm 1)$ is a locally constant function. Therefore, $\{e, \hat{n}, \hat{b}\}$ and $\hat{\kappa}$ are called the *extended Frenet frame* and the *extended curvature function* of γ, respectively. We remark that $\hat{n}$, $\hat{b}$ and $\hat{\kappa}$ might change sign when γ is traversed once. In fact, we have

$$\hat{n}(s+l) = (-1)^N \hat{n}(s), \quad \hat{b}(s+l) = (-1)^N \hat{b}(s), \quad \hat{\kappa}(s+l) = (-1)^N \hat{\kappa}(s),$$

where N is the sum of the numbers of zeros of $\hat{\kappa}$ counted with multiplicity on $[0, l)$. Here, the *multiplicity* of a zero $s = c$ of a C^ω-function $\varphi = \varphi(s)$ is the non-negative integer μ satisfying

$$\varphi(c) = \varphi'(c) = \cdots = \varphi^{(\mu-1)}(c) = 0, \quad \varphi^{(\mu)}(c) \neq 0.$$

On the other hand, when $\kappa(s) \neq 0$, the function $\hat{\tau}(s) = -\langle D_s \hat{b}(s), \hat{n}(s) \rangle$ is equal to the torsion function $\tau(s) := -\langle D_s b(s), n(s) \rangle$ of γ. So we do not distinguish τ and $\hat{\tau}$ in the case of type I.

We define the geometric twisting number of a strip as follows: Let $F_{\gamma,\xi}$ be a closed strip defined by (1). We normalize the ruling vector $\xi(s)$ for each $s \in \mathbb{R}$ so that the projection of $\xi(s)$ into the normal plane at the point $\gamma(s)$ is a unit vector, i.e.

$$|e(s) \times \xi(s)| = 1 \quad (s \in \mathbb{R}), \tag{5}$$

where the *normal plane* is the plane perpendicular to $e(s)$. The vector field $\xi(s)$ can be expressed as a linear combination of $\{e, \hat{n}, \hat{b}\}$ by

$$\xi(s) = p(s)e(s) + \cos\theta(s)\hat{n}(s) + \sin\theta(s)\hat{b}(s), \tag{6}$$

where $p(s)$ and $\theta(s)$ are C^ω-functions. There exists an integer m such that

$$p(s+l) = (-1)^{m+N} p(s), \quad \theta(s+l) = \theta(s) + m\pi. \tag{7}$$

The number

$$\mathrm{Gtn}(F_{\gamma,\xi}) := m/2 \tag{8}$$

is called the *geometric twisting number* of $F_{\gamma,\xi}$ in Ω_γ of type I.

PROOF of Theorem A. A necessary and sufficient condition for the developability of a strip can be expressed by the two functions $p(s)$, $\theta(s)$ in (6). In fact, since

$$\gamma' \wedge \xi \wedge D_s \xi = (\theta' - p\hat{\kappa}\sin\theta + \hat{\tau}) e \wedge \hat{n} \wedge \hat{b}$$

holds, the developability of $F_{\gamma,\xi}$ is equivalent to

$$\theta' = p\hat{\kappa}\sin\theta - \hat{\tau}. \tag{9}$$

Therefore, the existence and non-existence of twisting numbers of developable closed strips along γ can be reduced to those of C^ω-functions $p(s)$, $\theta(s)$ satisfying (7) and (9). Consequently, since the condition (9) coincides with the equality (2.8) in [2], the same discussion as in [2] can be applied to Ω_γ of type I. □

3. The case of type II

Suppose that Ω_γ is non-orientable, i.e. of type II. Considering a sufficiently small open tubular neighborhood $\tilde{\Omega}_\gamma$ of the double covering of γ, we can define a (positive) orthonormal C^ω-frame $\{\boldsymbol{e}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$ and two C^ω-functions $\hat{\kappa}$, $\hat{\tau}$ satisfying (4) in the orientable $\tilde{\Omega}_\gamma$. By the identification of the tangent spaces $T_{\gamma(s+l)}M^3$ and $T_{\gamma(s)}M^3$ for each $s \in \mathbb{R}$ via the covering map $\rho\colon \tilde{\Omega}_\gamma \to \Omega_\gamma$, we can obtain the orthonormal C^ω-frame $\{\boldsymbol{e}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$ in the non-orientable Ω_γ as follows;

$$\hat{\boldsymbol{n}} := \hat{\boldsymbol{b}} \times_\gamma \boldsymbol{e}, \quad \hat{\boldsymbol{b}} := \frac{\psi}{|\psi|}, \quad \psi(s) := \gamma'(s) \times_{\gamma(s)} D_s\gamma'(s) \left(\prod_{h=1}^{r} \sin^{N_h} \frac{(s - s_h)\pi}{l} \right)^{-1},$$

where $\times_{\gamma(s)}$ is the projection of the vector product at $\gamma(s)$ in $\tilde{\Omega}_\gamma$ (i.e. $A \times_{\gamma(s+l)} B = -A \times_{\gamma(s)} B$ for $s \in \mathbb{R}$ and $A, B \in T_{\gamma(s)}M^3$), the points $s = s_1, s_2, \ldots, s_r$ are all zeros of $\hat{\kappa} = \langle D_s\boldsymbol{e}, \hat{\boldsymbol{n}} \rangle$ on $[0, l)$ and $N_1, N_2, \ldots, N_r$ are the multiplicities of their zeros. We remark that the vector field ψ is well-defined on $\mathbb{R}$ and does not vanish. In this case, since

$$\hat{\boldsymbol{n}}(s + l) = (-1)^N \hat{\boldsymbol{n}}(s), \quad \hat{\boldsymbol{b}}(s + l) = (-1)^{N+1} \hat{\boldsymbol{b}}(s), \quad \hat{\kappa}(s + l) = (-1)^N \hat{\kappa}(s),$$

where $N := N_1 + N_2 + \cdots + N_r$, the extended torsion function $\hat{\tau} := -\langle D_s\hat{\boldsymbol{b}}, \hat{\boldsymbol{n}} \rangle$ changes sign when γ is traversed once. If $\hat{\kappa} \not\equiv 0$, then $\hat{\tau}$ is equal to the torsion function $\tau := -\langle D_s\boldsymbol{b}, \boldsymbol{n} \rangle$ up to sign, where $\{\boldsymbol{e}, \boldsymbol{n}, \boldsymbol{b}\}$ is the projection of the Frenet frame of γ via $\rho\colon \tilde{\Omega}_\gamma \to \Omega_\gamma$.

Under the normalization (5), the ruling vector field ξ of a closed strip $F_{\gamma,\xi}$ can be expressed by (6) as a linear combination of $\{\boldsymbol{e}, \hat{\boldsymbol{n}}, \hat{\boldsymbol{b}}\}$. Since $\xi(s+l) = (-1)^\lambda \xi(s)$ for some $\lambda \in \mathbb{Z}/2\mathbb{Z}$, we have $p(s + l) = (-1)^\lambda p(s)$ and

$$\cos\theta(s + l) = (-1)^{\lambda+N} \cos\theta(s), \quad \sin\theta(s + l) = (-1)^{\lambda+N+1} \sin\theta(s),$$

which is equivalent to

$$p(s + l) = (-1)^{m+N} p(s), \quad \theta(s + l) = -\theta(s) + m\pi \tag{10}$$

for some $m \in \mathbb{Z}$ satisfying $\lambda \equiv m + N \pmod 2$. If necessary, we replace $\theta(s)$ by $\breve{\theta}(s) := \theta(s) - \lfloor m/2 \rfloor \pi$ in (6), where $\lfloor\ \rfloor$ is the floor function, and then the corresponding m in (10) takes values in $\mathbb{Z}/2\mathbb{Z}$. Therefore, the *geometric twisting number* of $F_{\gamma,\xi}$ should be defined by

$$\mathrm{Gtn}(F_{\gamma,\xi}) := \begin{cases} 0 & (\text{if } m \text{ is an odd number}), \\ 1/2 & (\text{if } m \text{ is an even number}). \end{cases}$$

REMARK 3.1. If $m = 1$ (i.e. $\mathrm{Gtn}(F_{\gamma,\xi}) = 0$) and $\hat{\kappa} \neq 0$, then $\xi(s + l) = -\xi(s)$ holds, i.e. the strip $F_{\gamma,\xi}$ is non-orientable. The isotopy type of a closed strip along γ in Ω_γ of type II is determined by only the orientability of the strip.

Since the developability condition of $F_{\gamma,\xi}$ is also equivalent to (9), we can prove Theorem B in the same manner as Section 2, except for the difference between (7) and (10). Moreover, this difference can be resolved by the following lemma, which is analogous to [2, Lemma A.4]:

LEMMA 3.2. *Let $\varphi(t)$ be a 2π-odd-periodic C^∞-function. Let $t_1, t_2, \ldots, t_r$ be r points satisfying $0 \leq t_1 < t_2 < \cdots < t_r < 2\pi$ and $\nu_1, \ldots, \nu_r$ be positive integers. Then, there exists a sequence $\{\varphi_q\}_{q=1}^\infty$ of 2π-odd-periodic C^ω-functions such that*

$$j^{\nu_i - 1}\varphi_q(t_i) = j^{\nu_i - 1}\varphi(t_i) \quad (1 \leq i \leq r)$$

for each $q \in \mathbb{N}$ and $\{\varphi_q^{(k)}\}_{q=1}^\infty$ converges uniformly to $\varphi^{(k)}$ on $\mathbb{R}$ for each $k \in \mathbb{N}$, where

$$j^\nu\varphi(t) := (\varphi(t), \varphi'(t), \ldots, \varphi^{(\nu)}(t))$$

for a non-negative integer ν.

PROOF. The case that $\varphi(t)$ is 2π-periodic was proven in [2, Lemma A.4]. We can prove this assertion by using a Chebyshev space $\mathcal{A}$ whose order is an odd number, rather than one with even order in the proof of [2, Lemma A.4].

PROOF of Theorem B. Suppose that $\hat{\tau}(s)$ vanishes identically on $\mathbb{S}^1$. If we take $p(s) := \theta(s) := 0$ (i.e. $\xi(s) := \hat{\boldsymbol{n}}(s)$) for $s \in \mathbb{R}$, then the conditions (9) and (10) for $m = 0$ are satisfied. On the other hand, if we take $p(s) := 0$ and $\theta(s) := \pi/2$ (i.e. $\xi(s) := \hat{\boldsymbol{b}}(s)$), then (9) and (10) for $m = 1$ hold. Therefore, for $n \in \{0, 1/2\}$ there exists a developable C^ω-strip $F_{\gamma,\xi}$ along γ such that $\mathrm{Gtn}(F_{\gamma,\xi}) = n$ when $\hat{\tau}(s)$ vanishes.

Next, suppose that there exists a point $s = s_0 \in [0, l)$ such that $\hat{\tau}(s_0) \neq 0$. If necessary, by the parallel shift $s \mapsto s + l$ of parameter, we may assume $\hat{\tau}(s_0) < 0$.

We construct a developable strip $F_{\gamma,\xi}$ along γ such that $\mathrm{Gtn}(F_{\gamma,\xi}) = 1/2$, i.e. $m=0$. Let $s_1 < \cdots < s_r$ be all the zero points of $\kappa(s)$ on $[0,l)$, and $N_1,\ldots,N_r$ their multiplicities. Without loss of generality, we may assume that s_0 is not equal to $s_1,\ldots,s_r$. As in the proof of [2, Lemma 4.1], we can take a C^ω-function $\theta(s)$ satisfying the following conditions:

(a) $\theta(s+l) = -\theta(s) \quad (s \in \mathbb{R})$,
(b) $\theta^{(k+1)}(s_h) = -\hat{\tau}^{(k)}(s_h)$ $(1 \leq h \leq r,\ 1 \leq k \leq N_h - 1)$,
(c) $\theta(s_0) = 0$, $\theta'(s_0) = -\hat{\tau}(s_0)$,
(d) $-\pi < \theta(s) < 0$ for $0 \leq s < s_0$, and $0 < \theta(s) < \pi$ for $s_0 < s < l$.

In fact, one can choose a C^∞-function $\theta(s)$ satisfying (a)–(d), and we denote by $\tilde{\theta}(s)$ such an odd-periodic C^∞-function $\theta(s)$. By Lemma 3.2, there exists a sequence $\{\theta_q\}_{q=1}^\infty$ of odd-periodic C^ω-functions satisfying (a)–(c) such that $\{\theta_q\}_{q=1}^\infty$ and $\{\theta_q'\}_{q=1}^\infty$ uniformly converge to $\tilde{\theta}(s)$ and $\tilde{\theta}'(s)$, respectively. Moreover, there exists a sufficiently large positive integer $q = q_0$ such that θ_{q_0} satisfies (d). We set $\theta := \theta_{q_0}$ and

$$p(s) := \frac{\theta'(s) - \hat{\tau}(s)}{\hat{\kappa}(s)\sin\theta(s)}.$$

By the conditions (b) and (c), the function $p(s)$ is real-analytic at $s = s_0, s_1, \ldots, s_r$, so it is well-defined on $\mathbb{R}$. The pair of $p(s)$ and $\theta(s)$ satisfies (9) and (10) for $m=0$. Therefore, we can conclude the case of $\mathrm{Gtn}(F_{\gamma,\xi})=1/2$. The case of $\mathrm{Gtn}(F_{\gamma,\xi})=0$ (i.e. $m=1$) is also similar. In fact, it suffices to construct a C^ω-function $\theta(s)$ satisfying

(a′) $\theta(s+l) = -\theta(s) + \pi \ (s \in \mathbb{R})$,
(b′) $\theta^{(k+1)}(s_h) = -\hat{\tau}^{(k)}(s_h)$ $(1 \leq h \leq r,\ 1 \leq k \leq N_h - 1)$,
(d′) $0 < \theta(s) < \pi$ for $s \in \mathbb{R}$,

instead of the conditions (a)–(d). The condition corresponding to (c) is omitted because of (d′). □

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References

[1] C. Chicone and N.J. Kalton, Flat embeddings of the Möbius strip in $\mathbb{R}^3$, Comm. Appl. Nonlinear Anal. **9** (2002), 31–50.
[2] K. Naokawa, Extrinsically flat Möbius strips on given knots in 3-dimensional spaceforms, Tohoku Math. J. **65** (2013), 341–356.

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